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SETS, SENTENCES, AND SYSTEMS. HANDBOOK FOR JUNIOR HIGH SCHOOL MATHEMATICS WORKSHOPS.

BY- VAN SPEYBROECK, JAMES   ROBBINS, MORTON
ILLINOIS STATE OFF. SUPT. PUB. INSTR., SPRINGFIELD

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THIS WORKBOOK FOR TEACHERS IS CONCERNED WITH IDEAS AND CONCEPTS THAT WERE CONSIDERED IN A JUNIOR HIGH SCHOOL MATHEMATICS PROGRAM. THE ORGANIZATION WAS DETERMINED BY TWO MAJOR GOALS--(1) TO PROVIDE AN INSERVICE TRAINING WORKSHOP WHICH WOULD BE OF IMMEDIATE USE TO THE JUNIOR HIGH SCHOOL MATHEMATICS TEACHER, AND (2) TO PROVIDE THE TEACHER WITH AN OVERVIEW OF THE MAJOR OBJECTIVES OF A JUNIOR HIGH SCHOOL MATHEMATICS PROGRAM AND THE THEORY UPON WHICH THESE OBJECTIVES ARE BASED. PRELIMINARY NOTIONS OF ELEMENTARY SET THEORY AND THE MATHEMATICAL SENTENCE ARE DISCUSSED BEFORE PROCEEDING TO THE NATURAL NUMBERS, THE INTEGERS, AND THE RATIONAL NUMBER SYSTEM. (RP)

Math

SETS, SENTENCES, AND SYSTEMS

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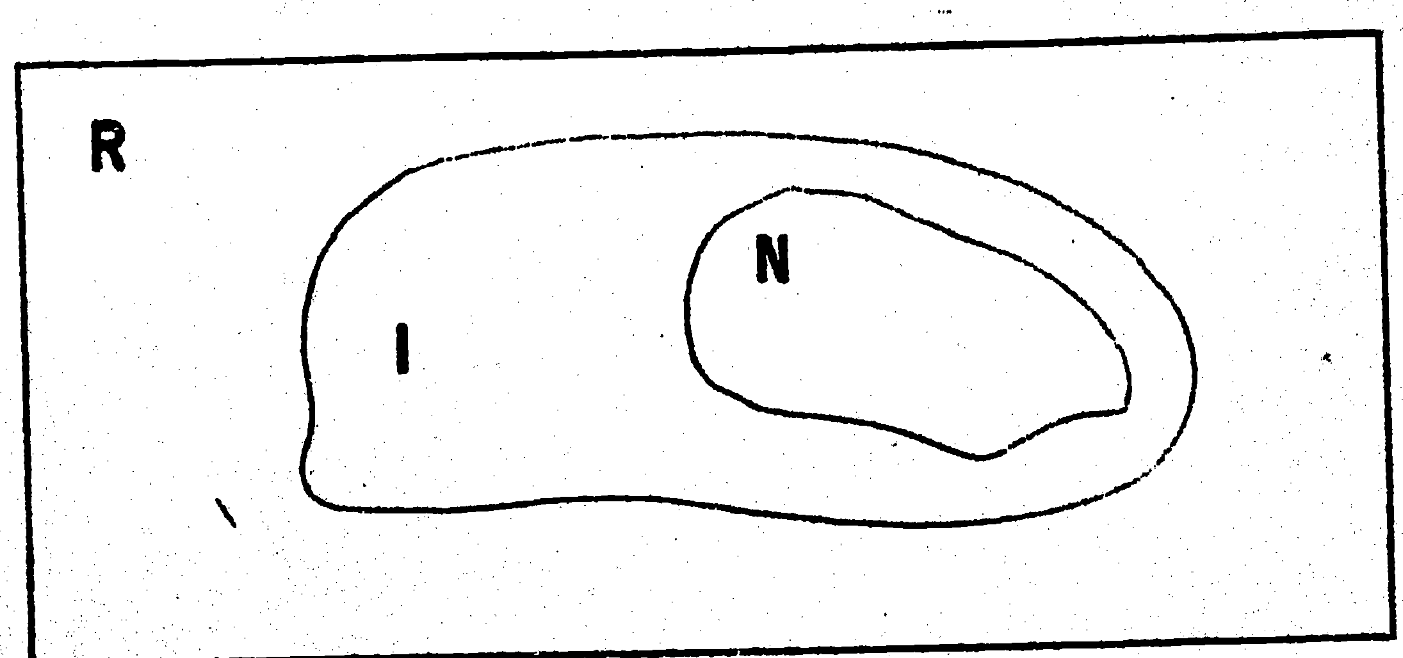
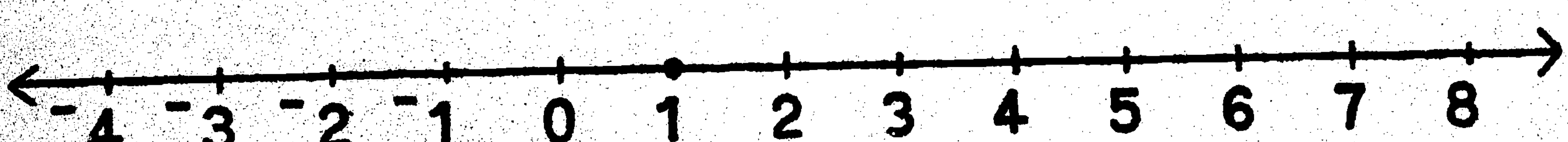
$N = \{0, 1, 2, 3, 4, 5, \ldots\}$

$I = \{0, 1, {}^{-}1, 2, {}^{-}2, 3, {}^{-}3, \ldots\}$

$R = \{a/b: a \in I, b \in I, b \neq 0\}$



$N \subset I$



$2x + 6 = 8$

ISSUED BY THE
OFFICE OF THE SUPERINTENDENT OF
PUBLIC INSTRUCTION

HANDBOOK

for

JUNIOR HIGH SCHOOL

MATHEMATICS WORKSHOPS

STATE OF ILLINOIS

The Office of the Superintendent of

Public Instruction

RAY PAGE, Superintendent

In recognition and appreciation of the unlimited time and effort given so generously and tirelessly for the betterment of mathematics education, this handbook is dedicated to

PAUL E. WOODS

His constant encouragement and continuous assistance has made the publication a reality.

The committee responsible for administering the pilot workshop held in Rockford, Illinois, in the spring of 1966, and for supervising the writing of this handbook consisted of

George L. Henderson and Joseph P. Cech,

mathematics consultants, Title III, NDEA, Office of Ray Page, State Superintendent of Public Instruction.

Authors of the handbook, who served as teachers for the pilot workshop, are James Van Speybroeck, Mathematics Department, Marycrest College, and Morton Robbins, Mathematics Department, Old Orchard Junior High School, Skokie, Illinois.

Valuable suggestions were received from Richard Meckes, James Mitchell, and Miss Frances Hewitt, Mathematics Consultants, Title III, NDEA, Office of Ray Page, State Superintendent of Public Instruction.

INTRODUCTION

This workshop will deal with the ideas and concepts vital to any junior high school mathematics program. The organization was determined by two major goals: (1) To provide an in-service training workshop which would be of immediate use to the junior high school mathematics teacher, and (2) to provide the teacher with an overview of the major objectives of a junior high school mathematics program and the theory upon which these objectives are based.

This last objective is an especially important one. If the teacher of today's mathematics does not understand how those principles of mathematics which are taught fit into the broad objectives of mathematics education, that teacher will be sharply limited in the accomplishment of his task.

To these ends "Sets, Sentences, and Systems" was written. It is hoped that this workshop will be both an enjoyable and profitable experience for you.

TABLE OF CONTENTS

UNIT I

SET THEORY

INTRODUCTION

What is meant in mathematics by the term, "sets"? What is the role of the "set" in the present development of mathematics? Questions such as these are unavoidable in a time when mathematics education is undergoing such revolutionary changes. It is quite possible for two mathematicians to spend their entire lives studying mathematics and yet not understand what the other is doing.

In order to meet the challenge of teaching mathematics for an age when mathematics knowledge is increasing in almost astronomical proportions, new courses have been devised and along with this, the more standard mathematics courses have been brought up to date.

One means by which mathematics has been unified and brought up to date is the innovation of Set Theory. An understanding of sets is a foundation upon which many concepts of mathematics are developed.

THEORY OF SETS

PRELIMINARY NOTIONS

SET It is not possible to give a definitive explanation of the word "set", but we shall think of a set as a well-defined collection of objects. Each object in a set is called an element of the set or, a member of the set.

Example: The number 2 is a member of the set of natural numbers. In symbols the small case greek letter "epsilon", ($\in$), means "is a member of."

SET DESIGNATION Generally speaking, a set can be designated in three different ways. However, it must be remembered that a set must be well-defined. This means that only two possibilities can exist; some object (1) is a member (in symbols, "$\in$", or (2) is not a member (in symbols, "$\notin$") of any given set. The set designation must be so clear as to specify all possible elements or members of a particular set.

Three ways will be considered in designating sets:

(1) A set may be designated by a statement.
Example: A is the set of all natural numbers between 1 and 5.

(2) A set may be designated by the Listing Method.
Example: $A = \{2, 3, 4\}$. The "braces" you see enclosing the numerals are used to indicate a set.

(3) A set may be designated by the Rule Method or, as it is sometimes called, the Conditional Method.
Example: $A = \{x : 1 < x < 5,\ x \in N\}$. The colon is read "such that." The symbol ">" is read "greater than," the symbol "<" is read "less than," and let N be the set of natural numbers. The example in its entirety would be read "A is the set of all x such that each x is a natural number greater than 1 and less than 5."

<u>SPECIAL SETS</u> In order to avoid certain dilemmas that might arise later on, two special sets will be defined.

(1) <u>Empty Set</u> The Empty Set is a set which contains no elements. This set is sometimes called <u>Null Set</u>, <u>Void Set</u>, or <u>Vacuous Set</u>. The symbol for the empty set is the greek letter "phi" ($\emptyset$), or { }.

<u>Example:</u> The set of all people who have three heads. Since this set would have no members, it would be designated as an empty set.

(2) <u>Universal Set</u> In adopting the <u>Rule Method</u> of designating sets, we sometimes specify that only certain elements may be used as replacements for the variable "x". When this is done, a special set is invented called the <u>Universal Set</u>. Only from this special set can the replacements for "x" be obtained. The Universal Set is sometimes called the "Replacement Set".

<u>Example:</u> $A = \{x : 1 < x < 5,\ x \in N\}$. In this instance (or for this discussion) the universal set is the set of natural numbers. Thus 2 1/2 cannot be considered as a replacement for x.

<u>SET RELATIONS</u> When two sets from the same Universal Set are part of a discussion, we may find that the sets have some or all or no members in common. In set theory, we have specific terminology to describe such situations.

<u>SUBSET (INCLUSION)</u> Some set (call the set "Set A") is a <u>subset</u> of some other set (call this set "Set D") if every element of set A is also an element of set D. In symbols, we write $A \subseteq D$. A careful application of the definition would indicate that $D \subseteq D$ and $\emptyset \subseteq D$.

<u>PROPER SUBSET (PROPER INCLUSION)</u> Some set A is a proper subset of some set D if, (1) A is a subset of D, and (2) there is at least one element of D which is not in A. In symbols we write $A \subset D$, which is read, "A is a proper subset of D".

<u>Examples:</u> (1) Consider a universal set which consists of the letters of the alphabet, i.e., $U = \{a, b, c \ldots x, y, z\}$. Consider these sets: $G = \{t, r, y\}$ $K = \{e, r. u\}$ $P = \{u, r, e, q, f\}$ $Y = \{e, r, u\}$. According to the definitions G, K, P, and Y are all proper subsets of the universal set.

Exercise: Using the universal set given in example 1 and the subsets given in the same example, state if the following are true or false. Also, explain why they are true or false.

(a) $G \subseteq U$
(b) $G \not\subset K$
(c) $Y \subset P$
(d) $K \subseteq P$ and $K \subset P$

Exercise (answers)

(a) True. $G \subseteq U$ since every element of G is an element of U.
(b) True. $G \not\subset K$ since the elements of G are not elements of K.
(c) True. Y is a proper subset of P since every element of Y is an element of P and P contains at least one element which is not an element of Y.
(d) True. K is a subset of P since every element of K is an element of P. Also, K is a proper subset of P since P contains at least one element which is not an element of Y.

EQUAL SETS (IDENTITY) Two sets are identical if the two sets contain exactly the same elements. If two sets are identical, they are referred to as the same set, equal sets or identical sets. In symbols we write $A = D$. If two sets are not equal we write $A \neq D$.

There is an interesting correlation between the preceding relations of subsets and equal sets when one considers that for two sets to be equal, every element of set A must be an element of set D; and every element of set D must be an element of set A. Therefore, it can be stated that : $A = D$ if and only if $A \subseteq D$ and $D \subseteq A$.

Exercises:

(1) Describe the following sets in words:
 (a) $\{2, 4, 6, 8\}$
 (b) $\{1, 3, 5, 7\}$
 (c) $\{0, 3, 6, 9, 12, 15, 18\}$
 (d) $\{a, e, i, o, u, y\}$
(2) Use the listing method to describe the following sets:
 (a) The set of all letters in the alphabet used in spelling the word "FOLLOW".
 (b) The set of all consonants in the alphabet.
 (c) The set all people in the world who are 500 years old.
(3) If $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, tell if the following are true or false:
 (a) $\{4, 5, 7\} \subset U$
 (b) The set of whole numbers greater than or equal to 0 and less than 10 is equal to the set U.

(c) $\{x : x \text{ is positive}\} \subset U$
(d) $\{2, 3, 4\} = \{4, 3, 2\}$
(e) $\{1, 3, 5, 7\} \subset U$

Exercises (Answers):

(1) a) The set of all even numbers greater than 0 and less than 10.
b) The set of odd numbers greater than 0 and less than 9.
c) The set of all multiples of 3, from 0 to 18 inclusive.
d) The set of vowels in the English Alphabet.

(2) $\{F, O, L, W\}$
b) $\{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, x, z\}$
c) $\emptyset$

(3) a) True. 4, 5, 7 are all members of U, and U has members not contained in $\{4, 5, 7\}$.
b) True.
c) False. 10 is positive and not a member of U.
d) True. Changing the order does not change the set.
e) True. All elements of the set are members of U, and U contains at least one member which is not in $\{1, 3, 5, 7\}$.

SET OPERATIONS The next step in the discussion of set theory is to state some actions or operations which can be performed on any two sets. Some of these operations have similarities to operations of Arithmetic.

SET INTERSECTION Frequently in set theory the case arises where two sets contain elements which are the same. These elements can be referred to in a variety of ways, e.g., "elements common to the sets". The definition of set intersection is as follows:

DEFINITION The intersection of two sets (call these sets A and D) is the set of all elements which are in set A and also in set D. Using symbols, $A \cap D$ is read "the intersection of A and D". Intersection is a relation involving two sets. Thus it is referred to as a binary operation. Using the rule method to name a set, we have $A \cap B = \{x : x \in A \text{ and } x \in B\}$. Example: If $K = \{2, 3, 4\}$ and $L = \{4, 5, 6\}$ then $K \cap L = \{4\}$. Two sets are referred to as DISJOINT if their intersection is the empty set.
Let $F = \{3, 4, 5\}$ and $B = \{10, 100, 200\}$ then $A \cap B = \{\ \}$ or $\emptyset$.

Exercises Find the intersections of the following sets:

(a) If R={q, w, r} and I={o, r, t, v} then R ∩ I=
(b) If Y={5, 6, 7} and H={7, 8, 9} then Y ∩ H=
(c) If M={#, $, %, ¢} and D={(, $,#, ', 6} then M ∩ D=
(d) What would be the intersection of the set of even numbers and the set of odd numbers?
(e) If the intersection of two sets is the empty set, what conclusion can always be made about the two sets?

Exercises (Answers)
(a) R ∩ I= {r, w}
(b) Y ∩ H= {6. 7}
(c) M ∩ D= {#, $}
(d) The intersection of the set of even numbers and the set of odd numbers is the empty set since there are no elements common to both.
(e) If two sets are intersected and the result is the empty set, one can conclude that the sets must be disjoint.

SET UNION

The next operation is reminiscent of the operation of addition in arithmetic. By forming a set of elements consisting of all the elements in two given sets, we have united the two sets.

DEFINITION

The union of two sets A and D is the set of all elements which are in the set A or in the set D, or in both sets A and B. In symbols, A ∪ D means "the union of A and D". As in the case of intersection, union is a relation involving two sets, and thus is a binary operation. Using the rule method to name a set we have $A \cup B=\{x : x \in A \text{ or } x \in B\}$.

Examples: If A={a, c, g} and D={s, t, y} then A ∪ D={a, c, g, s, t, y}.
If K={1, 3, 5, 7} and L={1, 3, 9, 11} then K ∪ L={1, 3, 5, 7, 9, 11}.
Note that the elements "1" and "3" appear in both sets K and L. However, by definition these elements appear in the set union only once.

SET COMPLEMENTATION If Z is a subset of some universal set U, then the complement of Z with respect to U is the set of all elements in U which are not in Z. This set is also called the complement of Z. In symbols, "the complement of Z" is written as : Z', $\overline{Z}$ or ~Z. It should be noted that Z' is an operation involving only one set. Thus complementation is a unary operation, as distinguished from intersection and union which are binary operations. Using the rule method to name a set, we have $Z' = \{x : x \in U \text{ and } x \notin Z\}$.

Examples: Let the universal set be:

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ $L = \{5, 6, 7, 8\}$ $M = \{2, 5, 7, 9\}$ $G = \{3, 6, 9\}$, then

(a) $L' = \{1, 2, 3, 4, 9, 10\}$

(b) $M' = \{1, 3, 4, 6, 8, 10\}$

(c) $\overline{G} = \{1, 2, 4, 5, 7, 8, 10\}$

(d) What set do you think is the complement of U?

RELATIVE COMPLEMENT If we wish to find the complement of a set in relation to another set (different from the universal set), we say we wish to find the relative complement of the set. The relative complement of a set Q with respect to set P is the set of all elements in P exclusive of the elements in Q. Using the rule method, this would be the set, $\{x : x \in P \text{ and } x \notin Q\}$. Note that it is not necessary for Q to be a subset of P.

Example: If $S = \{2, 4, 6, 8, 10\}$ and $G = \{1, 2, 3, 4, 5, 6, 7\}$, then the relative complement of S with respect to G is the set consisting of the elements 1, 3, 5, 7, and this would be written as : $G - S = \{1, 3, 5, 7\}$. "G - S" is read, "The relative complement of S with respect to G".

Exercises: (1) Given the universal set $U = \{1, 2, 3, 4, 5, \ldots\}$ and three subsets $K = \{2, 4, 6, 8\}$, $D = \{1, 3, 5, 7\}$, and $R = \{8, 9, 10\}$; perform the following operations:

(a) $K \cup D =$	(e) $K - D =$
(b) $K \cap D =$	(f) $K' =$
(c) $(R \cup D) \cup K =$	(g) $\overline{R} =$
(d) $R - K =$	(h) $\sim K =$

(2) Using the sets given in example 1, state whether the following are true or false. Also, give reasons to support your answers.

(a) $K \subseteq D$	(e) $U \cap \emptyset = \emptyset$
(b) $K \subset D$	(f) $R \cap K = R$
(c) $K \cup U = U$	(g) $(R \cap U) \cup R = R$
(d) $R \cap U = U$	(h) R and K are disjoint sets

Exercises (Answers):

(1) (a) $K \cup D = \{1, 2, 3, 4, 5, 6, 7\}$

(b) $K \cap D = \emptyset$

(c) $(R \cup D) \cup K = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

(d) $R - K = \{9, 10\}$

(e) $K - D \{2, 4, 6, 8\}$

(f) $D' = \{2, 4, 6, 8, 9, 10 \ldots\ldots\}$

(g) $\overline{R} = \{1, 2, 3, 4, 5, 6, 7, 11, 12, 13, \ldots\ldots\}$

(h) $\sim K = \{1, 3, 5, 7, 9, 10, 11, \ldots\ldots\ldots\}$

(2) (a) False. The elements of K are not contained in D.

(b) False.

(c) True. Since the operation of union states that elements common to the two sets are to be included in the union only once, the resulting set would be identical to U.

(d) False. Not all of the èlements in U are shared or also elements of R.

(e) True. Since there are no elements in common, the intersection must be the empty set.

(f) False. R includes the elements 9 and 10 which are not found in K.

(g) True. $R \cap U$ is equal to R and $R \cup R = R$

(h) False. The element "8" is an element of both sets.

CARTESIAN CROSS PRODUCT The next set operation resembles the arithmetic operation of multiplication. However, it is important to remember that operations in one system of mathematics need not have a place in other systems of mathematics.

DEFINITION Consider some finite set $F = \{2, 3, 4, 5\}$ and some other finite set $N = \{a, s\}$. The Cartesian Cross Product of F and N (sometimes called the "cross product" of F and N) is defined as being equal to the set of all ordered pairs formed by matching (or crossing) each element of set F with each element of set N. In symbols, the cross product of F and N is written " F x N ". Thus,

$F \times N = \{(2, a), (2, s), (3, a), (3, s), (4, a), (4, s), (5, a), (5, s)\}$

When performing this operation of matching each element of F with each element of N to form a set of ordered pairs, it should be noted that the phrase, "ordered pair" is used in the strictest sense. An ordered pair is first, a pair, because two elements are involved. Second, these pairs are constructed observing a strict order, i. e., the first element in each pair comes from the set F and the second element comes from set N. Any other formulation would violate the definition. Note that this is a binary operation. Using the rule method this would be the set, $\{(x, y) : x \in F \text{ and } Y \in N\}$.

VENN DIAGRAMS AND VERIFICATION

There is an interesting way of giving physical representations of operations on sets and relations of sets. This physical representation is referred to by the term "Venn Diagram". First there are some preliminary notions which are stated.

For no particular reason, other than clarification of an abstract idea, the universal set is portrayed by the points of a rectangle (usually, although other plane figures are sometimes used) plus the interior of the rectangle. Any subsets of U (in all but a few cases, proper subsets) are represented by the points of circles plus their interiors.

Although, Venn Diagrams show both relations of sets and operations on sets, it seems as though their major function is the latter, i.e., to give a vivid picture of set operations.

Example: Consider some universal set U and two proper subsets of U, denoted by Z and F. Let us further state that Z is a proper subset of F, i.e., $Z \subset F$. The problem will be to show the union of Z and F, i.e., $Z \cup F$.

Figure A

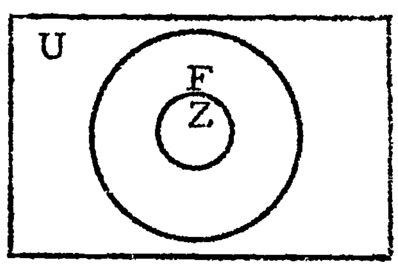


Figure B

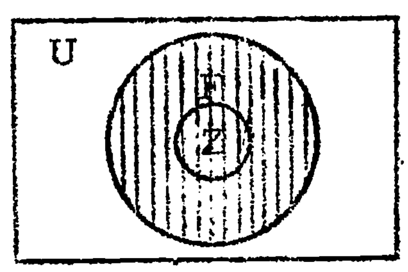
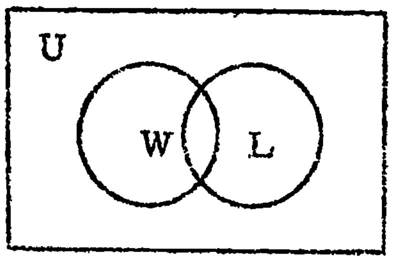


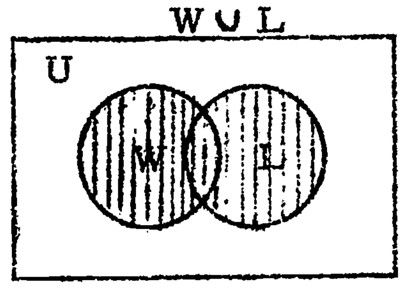
Figure A shows the relation of Z and F, while Figure B shows the union of Z and F most vividly by shading the area determined by $Z \cup F$.

Example: Consider some universal set U and two proper subsets of U, W. and L. However, unlike the first example, $W \not\subset L$ and $L \not\subset W$, but they do have some elements in common.

Problem: Show $W \cup L$.

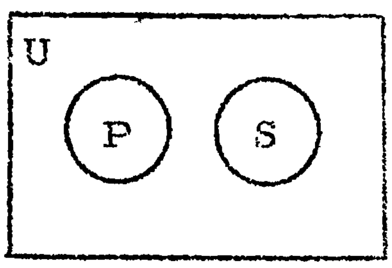


$W \cup L$

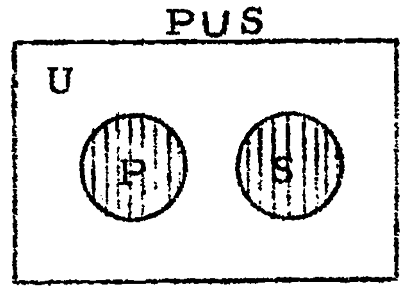


Example: Consider some universal set U and two proper subsets of U, P and S. Also, we define P and S to be Disjoint.

Problem: Show $P \cup S$.



$P \cup S$



Exercises:

(1) If $A=\{4, 5, 6, 7\}$ and $B=\{5, 6, 7, 9, 3\}$, use Venn Diagrams to show $A \cup B$.

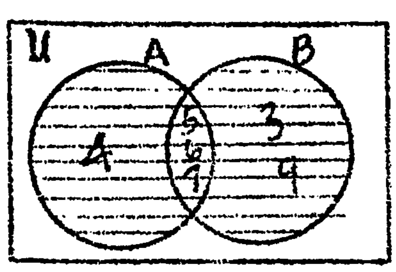
(2) Using the sets given in exercise 1 use Venn Diagrams to show that $A \not\subset B$.

(3) If $D=\{a, b, c, f\}$ and $H=\{f, c, a, b\}$ use Venn Diagrams to show D=H

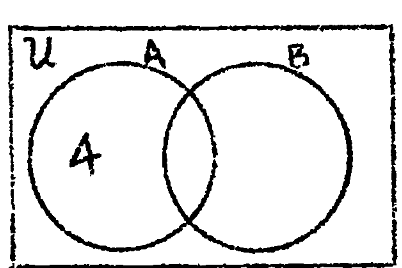
(4) Venn Diagrams are described as a means of verification but not a method of proof. Can you reason why this is so?

Exercises (Answers):

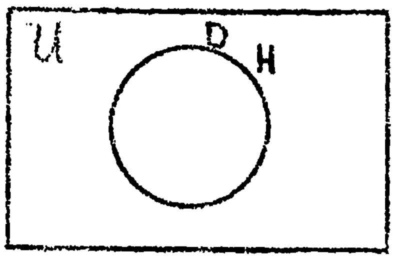
(1) Use Venn Diagrams to show $A \cup B$



(2) If A were a proper subset of B the Venn Diagram would place the set representing A inside the Venn Diagram B, but this is impossible since A contains some elements not in B.



(3) D = H The circles representing D and H are congruent.

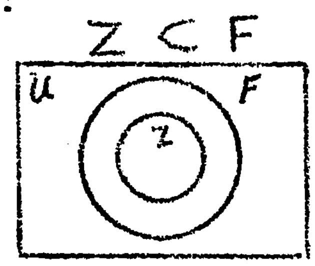


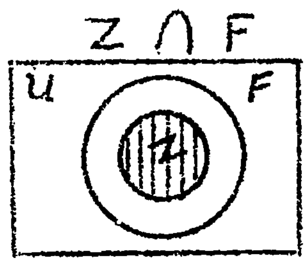
(4) Venn Diagrams are illustrations which show set operations and relations for specific cases rather than for generalizations. A proof is independent of specific cases and concerns itself with generalizations.

VENN DIAGRAMS AND SET INTERSECTION Venn Diagrams can be used to show the operation of set intersection. To show intersection we will place the same three conditions on the sets as we did in showing set union.

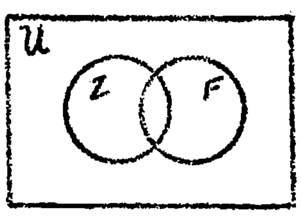
(1) Condition: $Z \subset F$. Problem: $Z \cap F$
(2) Condition: W and L contain some elements in common. Problem: $W \cap L$
(3) Condition: P and S are Disjoint. Problem: $P \cap S$

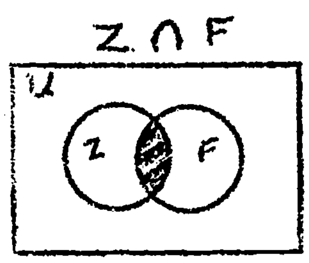
Example (1):



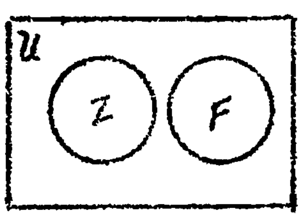


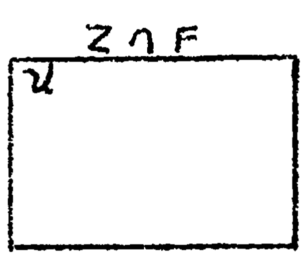
Example (2):





Example (3):



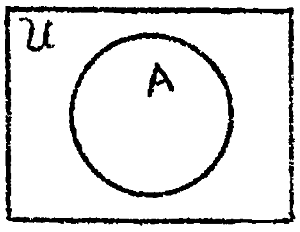


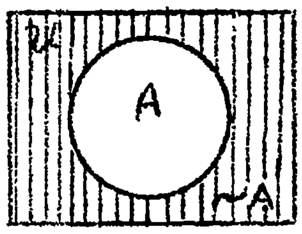
The representation of P S may cause some confusion. It will help to consider the following questions:

(1) What is the relationship of P and S?
(2) What is the intersection of two disjoint sets?

VENN DIAGRAMS AND COMPLEMENTATION Both the complement and relative complement operation can be shown vividly and easily through the use of Venn Diagrams.

Example: Consider a universal set U and some proper subset A. By definition, the complement of A with respect to U (in symbols, $\sim A$) equals to the set of all elements in U that are not in A. Using Venn Diagrams:

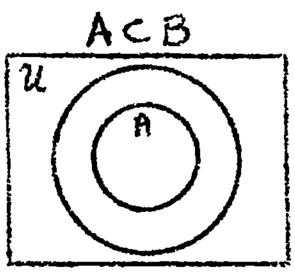


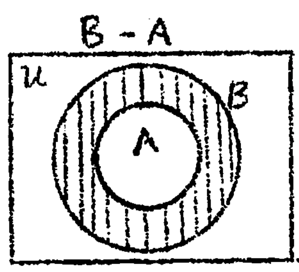


VENN DIAGRAMS AND RELATIVE COMPLEMENT Consider a universal set U and two proper subsets of U, A and B. By definition, the relative complement of A with respect to B is the set of all elements in B which are not in A. Consider the same conditions placed on the subsets that were placed on the subsets used in showing union and intersection.

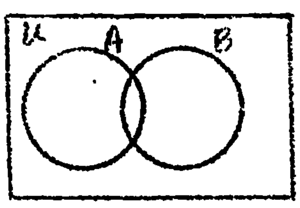
(1) Condition: $A \subset B$. Problem: to find B - A
(2) Condition: A and B share some elements.
Problem: to find B - A
(3) Condition: A and B are disjoint.
Problem: to find B - A

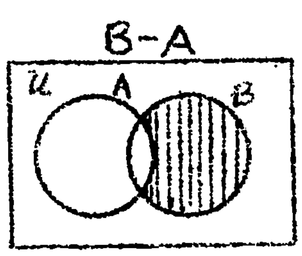
Example (1): If A B, then B - A would be pictured as shown:



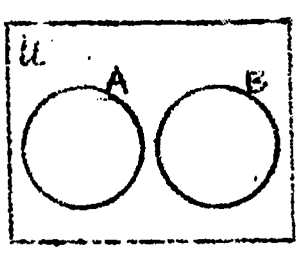


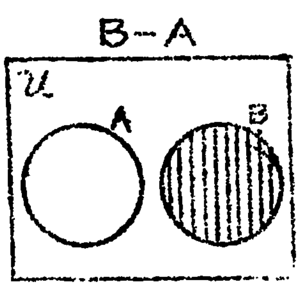
Example (2): If A shares some elements with B, then B - A would be as pictured below:





Example (3): If A and B are Disjoint, then B - A is represented as the following picture:





SET THEORY ASSIGNMENT

1. Which of the following sets are well-defined?
   a) The set of all people who have had the mumps.
   b) The set of all teachers.
   c) $A = \{0, 4, 8, 12, 16, 20\}$
   d) The set of all dogs who have a Ph. D. in mathematics.

2. Designate the following sets in at least one other way:
   a) The set of all multiples of 7.
   b) $A = \{1, 3, 5, 7, 9\}$
   c) $B = \{x : 1 < x < 5,\ x \in N\}$

3. $U = \{1, 2, 3, 4, 5, \ldots\ldots\}$, $A = \{1, 2, 3, \ldots. 10\}$, $B = \{1, 3, 5, 7, 9\}$
   $C = \{2, 4, 6, 8\}$ $D = \{3, 6, 9\}$ $E = \{5, 10\}$. Tell if the following are true or false and explain your answer.

   a) $A \subset U$
   b) $C \subseteq U$
   c) $C \subset U$
   d) $U \subseteq U$
   e) $U \not\subset U$
   f) $B \neq D$
   g) $D \not\subset B$
   h) $E \subset A \subset U$
   i) $\emptyset \subset U$
   j) $\emptyset \subseteq \emptyset$

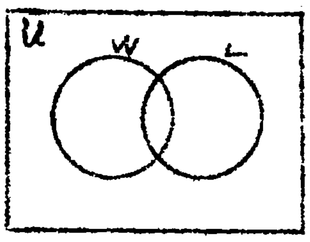
4. Using the sets given in problem (3), complete the following:

   a) $A \cup C =$
   b) $\sim E =$
   c) $(D \cap B) \cap U =$
   d) $\emptyset \cap U =$
   e) $E \times D =$
   f) $\emptyset \cup U =$
   g) $(A \cup C) \cap \emptyset =$
   h) $(A \cup B) \cap U =$
   i) $C - A =$
   j) $A - C =$

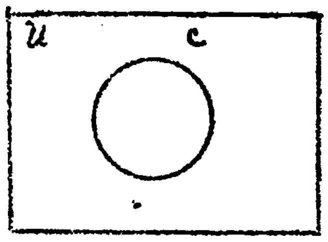
5. Using the sets given in problem (3), verify that the following are true:

   a) $A \cap C = C$
   b) $A \cup D = A$
   c) $(A \cap C) \cup A = A$
   d) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   e) $(A \cup D) \cup E = A \cup (D \cup E)$
   f) $A \cup \emptyset \neq \emptyset$
   g) $A \cap \emptyset = \emptyset$
   h) $A \cup U = U$
   i) $A \cap U = A$
   j) $D \times \emptyset = \emptyset$

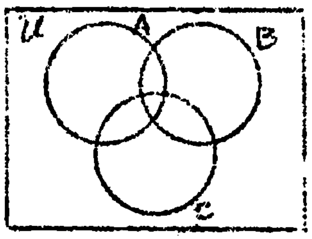
6. Using Venn Diagrams shade the area indicated by the following operations:



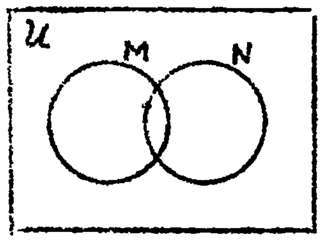
$\sim(\sim W)$



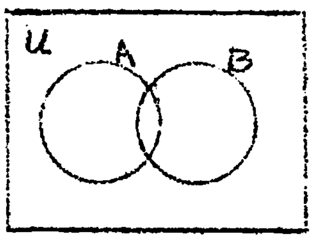
$U - C$



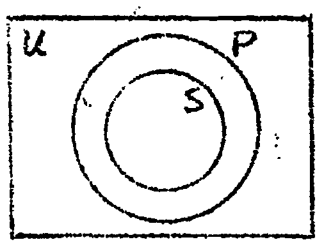
$(A \cap B) \cap C$



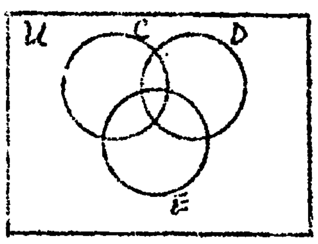
$\sim(M \cup N)$



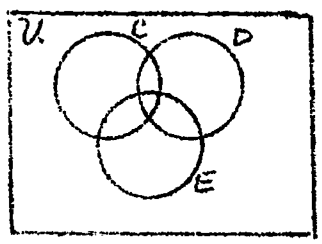
$(A - B) \cap A$



$P \cap S$



$(E \cap D) \cup C$



$(E \cup C) \cap (C \cup D)$

SET THEORY (ASSIGNMENT ANSWERS)

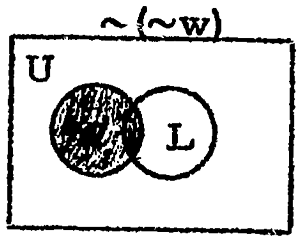
1. a) Not well-defined b) Well-defined (could be argumentative)
   c) Well-defined d) Well-defined (Empty set)
2. a) $\{0, 7, 14, 21, 28, \ldots\}$ b) The set of all odd natural numbers less than 10.
   c) $B = \{2, 3, 4\}$
3. a) True. A is a proper subset of U
   b) True. C is a subset of U
   c) True. C is also a proper subset of U
   d) True. U is a subset of U.
   e) True. U is not a proper subset of U
   f) True. B does not equal D
   g) True. D is not a proper subset of B
   h) True. E is a proper subset of A and A is a proper subset of U.
   i) True. The Empty Set by definition is a proper subset of every set.
   j) True. Every set is a subset of itself.

4. a) $\{1, 2, 3, \ldots 10\}$
   b) $\{1, 2, 3, 4, 6, 7, 8, 9, 11, \ldots\}$
   c) $\{3, 9\}$
   d) $\emptyset$
   e) $\{(5, 3), (5, 6), (5, 9), (10, 3), (10, 6), (10, 9)\}$
   f) U
   g) $\emptyset$
   h) $\{1, 2, 3, 4, \ldots 10\}$
   i) $\emptyset$
   j) $\{1, 3, 5, 7, 9\}$

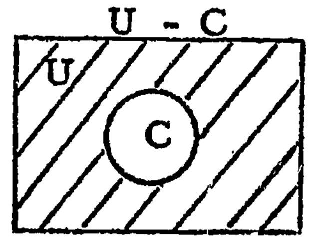
5. a) The element of C are in both A and C
   b) D is a subset of A
   c) $A \cap C = C$ and $C \cup A = A$
   d) $A \cap (B \cup C) = A \cap \{1, 2, \ldots, 8, 0\}$
      $= \{1, 2, \ldots, 8, 9\}$
      $(A \cap B) \cup (A \cap C) = B \cup C$
      $= \{1, 2, \ldots, 8, 9\}$
      $\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   e) $(A \cup D) \cup E = A \cup E = A$
      $A \cup (D \cup E) = A \cup \{3, 5, 6, 9, 10\} = A$
      $\therefore (A \cup D) \cup E = A \cup (D \cup E)$
   f) Since $A \cup \emptyset = A$,
      $A \cup \emptyset \neq \emptyset$ is true.
   g) Since there is no common element in both A and $\emptyset$,
      $A \cap \emptyset = \emptyset$ is true.
   h) Every element of U is in either A or U.
   i) Since every element in A is also in U, $A \cap \mathcal{U} = A$ is true.
   j) There is no element with which to pair the elements of D, therefore, the answer must be the empty set.

6.

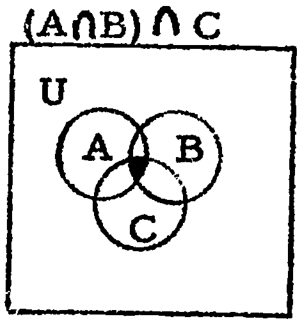
$\sim(\sim w)$



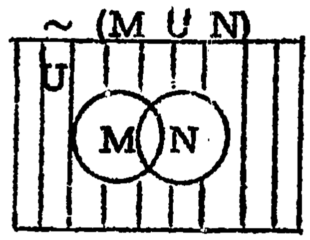
U - C



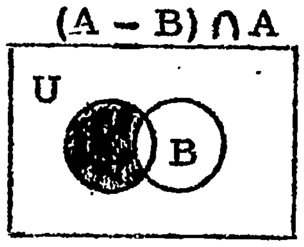
$(A \cap B) \cap C$



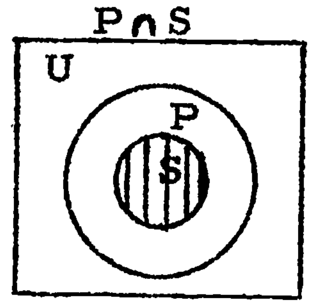
$\sim(M \cup N)$



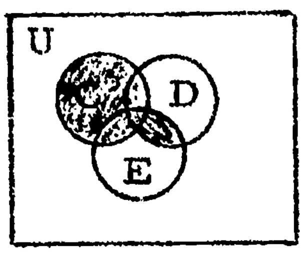
$(A - B) \cap A$



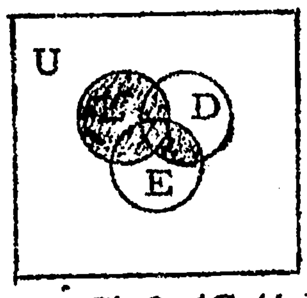
$P \cap S$



No. 6-Set Theory Assignment Answers - continued



$(E \cap D) \cup C$



$(E \cup C) \cap (C \cup D)$

THE MATHEMATICAL SENTENCE

In order to unify mathematics and correlate it with the logic of English, a term has been introduced which has gained more than a little acceptance by mathematics educators.

MATHEMATICAL SENTENCE A mathematical sentence is a proposition relating two mathematical expressions. "$2x = 6$" is a mathematical sentence; because it has a subject ($2x$), and object (6), and a predicate ($=$).

Exercises: Locate the subject, predicate, and object in the following mathematical sentences:

(1) $x - 9 = 14$
(2) $3x < 4$
(3) $4y > 19$
(4) $7x + 4y = 15$

That the term mathematical sentence is an innovation can be clearly seen, since formerly "$2x = 6$" would be classified solely as an equation. $2x = 6$ is still an equation, but the term sentence has been invented, because it is more readily understood by the beginning student of mathematics. Also, the use of mathematical sentence is a unifying idea in that it includes several relations formerly titled by a series of different terminology.

OPEN SENTENCE AND CLOSED SENTENCE

In mathematics when we cannot adjudge the truth of a mathematical sentence, the mathematical sentence is called an Open Sentence.

"$x = 7$" is an open sentence. If the placeholder "x" is replaced by the name of some natural number, the sentence is no longer open; it becomes a Closed Sentence.

TRUE STATEMENT AND FALSE STATEMENT

If a closed sentence is true, it is called a True Statement. If the closed sentence is untrue, it is called a False Statement.

Example: $x = 7$ Open Sentence

Replace the placeholder "x" by the natural number "6."
Then the result obtained is $6 = 7$.
Now the sentence is a Closed Sentence.
However, since $6 = 7$ is not true, we have a False Statement.
Replace the placeholder by the natural number "7."
Then the result obtained is $7 = 7$.
Thus we have a Closed Sentence and a True Statement.

Our objective will be to find all eligible replacements of the placeholder which make the sentence true. The set of all such replacements is called the solution set of the open sentence.

MATHEMATICAL REASONING

The history of the development of mathematics is a human story. It is a story of mistakes, blunders, and blind prejudice. It seems that the numbers 1, 2, 3, etc. are as obvious as night and day; and yet, the understanding of the concept of number took thousands of years. Even after man had formulated in his own mind the concept of number, it took him another thousand years to invent a symbol or numeral to express this idea of number; i.e., "1", "2", etc.

Today mathematics has reached a high level of development. Part of this development is due to that quality of mathematics called "flexibility." There is a freedom in mathematics which can be found in no other science. This freedom is a result of the total objectivity of mathematics.

Bertrand Russell, a contemporary philosopher-mathematician, has given this definition of mathematics: "Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true." Although this definition seems a bit ridiculous, Mr. Russell uses it to make a very important point; MATHEMATICS IS AN INVENTION, NOT A DISCOVERY.

In mathematics we are no more bound to the idea that $5 + 6 = 11$ than we are bound to the fact that we must live in only one house all of our lives.

Because mathematics is an invention, we may alter or change this invention any time we wish; just as the inventor changes from the candle as a form of illumination to the light bulb. The idea which concerns the mathematician is, "How do I start?" or, "What definitions do I make?" If the mathematician starts with the definition, "The moon is made of green cheese with a surface of peaches and cream," he can solve the following problems:

Question: "Is the moon good to eat?"

Answer: "Yes, if you like green cheese topped with peaches and cream."

Question: "Could we construct buildings with material found on the moon?"

Answer: "No (unless we wanted an edible, but very unstable building)."

The point this example stresses is, in mathematics certain definitions are stated. From these definitions any action may be performed as long as the action or operation agrees with the definitions.

In this workshop we are going to invent some mathematical systems. We will proceed much as the inventor proceeds.

First, we need Undefined Terms. These are ideas we wish to use, but cannot explain; just as the inventor, Thomas Edison, used electricity for his incandescent lamp, yet Mr. Edison could not explain electricity.

Examples of undefined terms would be the notions of "numbers" or, in geometry, the notion of a "point."

Next, these undefined terms are used to construct Definitions, just as the inventor uses electricity (which he cannot define) to make definitions; i.e., the definition of Voltage, Coulomb, Ampere, etc.

Using these definitions, rules or axioms will be stated that must be followed, just as Mr. Edison used rules or axioms such as, Voltage divided by Resistance equals current.

Finally, we will use these undefined terms, definitions, and axioms to solve problems or establish theorems, just as Mr. Edison used his light bulb to solve the problem of lighting the world in a better and more efficient manner.

Just as inventors through the years improved on Thomas Edison's light bulb to meet new responsibilities and challenges, we will improve or extend our mathematical system, so that it will handle more complex problem-solving situations.

Remember - To invent a mathematical system, we use a very definite approach (sometimes called the Axiomatic Approach).

We need:
(1) Undefined Terms
(2) Definitions (Based on the undefined terms)
(3) Axioms (Rules which must be followed)
(4) Theorems (conclusions derived from (1), (2), and (3) with the use of logic).

UNIT II

THE NATURAL NUMBERS

INTRODUCTORY DISCUSSION The familiar numbers which are used for counting are called, by mathematicians, natural numbers. Although it is customary to begin counting with the number one, in this workshop we will consider zero to be the first (smallest) natural number. It must be noted that there is considerable lack of agreement among mathematicians as to whether zero should be included in the set of natural numbers.

BUILDING THE SET OF NATURAL NUMBERS To "build" the set of natural numbers we will need some mathematical "tools" which will be discussed herein:

RELATIONS In the physical world objects are associated by relations such as "is the father of," "is older than," "is next to" and "is between" to name a few. In the set of natural numbers we will use the following relations:

EQUALITY This relation is denoted by the symbol "=" which is read "is equal to." The symbol "=" placed between two mathematical names indicates that both names refer to the exact same mathematical object. For example, if a and b are natural numbers, $a = b$ indicates that a and b are the exact same number.

In the current approach to the study of mathematics a careful distinction is made between "number" and "numeral." A number is a purely mental concept which is governed by various mathematical laws. A numeral is a name for a number and in its written form (as distinguished from spoken symbols) can be combined with other mathematical symbols and manipulated to convey mathematical ideas.

INEQUALITY For any two natural numbers a and b, "$a \neq b$" means that a is not equal to b. It should be noted that there is no way of determining, from such an expression, the relative sizes of the two numbers. All other inequality relations do indicate the relative sizes of numbers.

$a>b$: "a is greater (larger) than b"

$a<b$: "a is less (smaller) than b"

$a \geq b$: "a is greater than or equal to b"

$a \leq b$: "a is less than or equal to be"

$a \ngtr b$: "a is not greater than b"

$a \nless b$: "a is not less than b"

OPERATIONS All mathematical objects (of which number is just one type) are mental concepts and are dealt with by means of mental concepts. In the world of numbers, an operation is a process which generates numbers from other numbers. A binary operation generates numbers from two numbers, while a unary operation generates numbers from one number. In many current mathematics programs an operation is described as a process in which (for a binary operation) a pair of numbers is "mapped onto" (associated with) a unique number. A mapping of this type is sometimes indicated in the following way:

$$^{+}(1, 2) \longrightarrow 3$$

"Under the operation of addition the pair 1, 2 is associated with, and only with, the number three."

THE SET OF NATURAL NUMBERS Each natural number is associated with a particular set (the type of objects in the set is immaterial) and is said to denote the cardinality of that set. A natural number used in this way is called a cardinal number.

SET	NATURAL NUMBER
$\{\ \ \}$	Zero
$\{\triangle\}$	One
$\{\triangle, \square\}$	Two
$\vdots$	$\vdots$

If the members of two different sets can be put into a one to one correspondence so that each member of one set has a unique mate in the other set, the two sets are said to have the same cardinality. In attempting to make a one to one correspondence between the members of two different sets, if one set contains a single member which has no mate in the other set, the set which contains the "extra" member is said to be "one larger" than the other set. The number associated with the larger set is "one more" than the number associated with the smaller set.

The set of natural numbers is an ordered set. Each number, beginning with zero, has a unique successor which is one more. If we let N represent the set of natural numbers,

N = { 0, 0+1, (0+1)+1 and so on increasing by one without end}

COUNTING Counting is a process in which the members of a set are put into a one to one correspondence with an ordered sequence of the natural numbers beginning with the number one. The last number in the sequence denotes the cardinality of that set. Any set whose members can be put into such a correspondence with the natural numbers is called a countable set. This has no relation to the physical possibility of carrying out such a task (as for example with all the grains of sand on all the beaches on earth), but depends upon whether or not the set contains discrete objects which can be "lined up" in some order. If the counting comes to an end the set is said to be denumerably finite; and if the counting never ends, the set is said to be denumerably infinite.

The set of natural numbers is denumerably infinite, because it can be put into a one to one correspondence with itself without end. It is interesting to note that a denumerably infinite set can have many proper subsets which are also denumerably infinite; e. g., the set of even numbers and the set of odd numbers are denumerably infinite; and they are proper subsets of the set of natural numbers.

We will use the letter n, to the left of a symbol designating a particular set, to indicate the number of elements in that set. For example

$$n\{ a, b \} = 2$$

If two sets, A and B, contain exactly the same members we indicate this by the expression "A = B." If two sets, A and B, contain the same number of members, we indicate this by the expression "$A \sim B$" which means "A is equivalent to B." Sets which have the same cardinality are said to be equivalent sets.

EXERCISES:

a) $n\ \emptyset =$ _____	d) $n\{2, 3\} =$ _____
b) $n\{0\} =$ _____	e) $n\{(2, 3)\} =$ _____
c) $n\{2+3\} =$ _____	f) $n\{\emptyset\} =$ _____

2) For two sets, X and Y, if $X \sim Y$ is $X = Y$?

3) For two sets, R and S, if $R = S$ is $R \sim S$?

4) If $a \not> b$ then what must be true of b?

5) If $a \not< b$ then what must be true of b?

ANSWERS TO EXERCISES:

1) a) 0; b) 1; c) 1 because 2+3 is a name for number, five. d) 2; e) 1 because (2, 3) is a single element in the form of an ordered pair; f) 1 because the set contains one element which itself is a set; the fact that it is the empty set does not affect the answer which would be the same if a set were to contain one non-empty set as its only member; e. g., $n\{ \{ a, b, c\} \} = 1$.

- continued

2) Not necessarily. It is possible, but there is no way of telling from the given information.
3) Yes. If two different sets have exactly the same members, then they must have exactly the same number of members.
4) $b \geq a$
5) $b \geq a$

THE SYSTEM OF NATURAL NUMBERS Thus far we have discussed the natural numbers as a set of related mathematical elements. Now we will examine this set of numbers with regard to the operations which are defined on it, and in doing so we will be considering the natural numbers as a mathematical system.

MATHEMATICAL SYSTEM A mathematical system consists of a set of elements and at least one operation defined on that set. The results of combining elements of a set according to the defined operations are determined by laws which are either postulated (assumed) or which are derived from existing definitions and assumptions. These laws are often called properties.

In general it is educationally profitable for students to be encouraged, wherever possible, to discover or to postulate various properties as a result of their number experiences.

PROPERTIES OF THE NATURAL NUMBERS UNDER ADDITION

In the following discussion "N" will denote the set of natural numbers and all lower case letters will denote natural numbers.

1) For any a and b, $a + b \in N$. This is known as the closure property of addition, and means that the result of adding any two natural numbers is a natural number. The set of natural numbers is said to be closed under the operation of addition; because it is impossible to generate, by addition, a number which is not a member of the set.

2) For any a and b, $a + b = b + a$. This known as a commutative property of addition, and means that the sum of two natural numbers is unaffected by the order in which the numbers are combined. In future illustrations we will refer to this property as C.P.A.

3) For any a, b and c, $a + (b + c) = (a + b) + c$. This is known as the associative property of addition, and means that the sum of three natural numbers is unaffected by the grouping of the numbers. Since addition is a binary operation only two natural numbers can be combined at one time. We will refer to this property as A.P.A.

4) For any a, $a + 0 = a$. The number zero is called the identity element for addition in the set of natural numbers. Combining

any natural number with zero by the operation of addition produces that identical number. Stated in another way, "if $a + x = a$ then $x = 0$."

PROPERTIES OF EQUALITY

I. REFLEXIVE PROPERTY OF EQUALITY $a = a$
Any natural number is equal to itself.

II. SYMMETRIC PROPERTY OF EQUALITY If $a = b$, then $b = a$

III. TRANSITIVE PROPERTY OF EQUALITY If $a = b$ and if $b = c$, then $a = c$.

IV. ADDITION PROPERTY OF EQUALITY If $a = b$, then $a + c = b + c$.

SUBSTITUTION PROPERTY

Different names for the same mathematical object can be interchanged in mathematical expressions without changing the meaning of those expressions.

SUBTRACTION IN THE SET OF NATURAL NUMBERS We will define subtraction in reference to addition as follows:

If $a + b = c$, then $b = c - a$

If "b" is replaced in the first expression by its equivalent "c - a" the result is $a + (c - a) = c$. This leads to the following definition: "c - a is the number which when added to a produces c." By the same reasoning if $a + b = c$ then $a = c - b$ and $c - b$ is the number which when added to b produces c.

In the above context $c \geq a$ and $c \geq b$ since otherwise the symbols "c - a" and "c - b" would have no meaning in the set of natural numbers. The set of natural numbers is not closed under the operation of subtraction; e.g., $2 - 3$ is not a natural number.

GENERAL DEFINITION OF SUBTRACTION IN THE SET OF NATURAL NUMBERS $a - x = b \Longleftrightarrow a = b + x$ for $x \leq a$

The definition of subtraction can be combined with the identity element for addition to produce the following corollary:

$$n - n = 0$$

Proof:	Let $n - n = x$		
	n	$= x + n$	Definition of subtraction
	x	$= 0$	Definition of identity element for addition
	$n - n = 0$		Transitive property of equality

THE "GREATER THAN" RELATION

DEFINITION OF GREATER THAN If $a > b$, there exists some x such that $b + x = a$; where $x \neq 0$.

AN ADDITION PROPERTY OF ">" $a > b \longrightarrow a + c > b + c$ *

Proof:	$a > b$	Given
	$a = b + x$	Definition of >
	$a + c = (b + x) + c$	Addition property of equality

- continued

$a + c = b + (x + c)$ A.P.A.

$a + c = b + (c + x)$ C.P.A.

$a + c = (b + c) + x$ A.P.A.

$a + c > b + c$ Definition of >

*The converse will be proved as an exercise.

THE TRANSITIVE PROPERTY OF ">" If $a > b$ and if $b > c$, then $a > c$

Proof:	$a > b$	Given
	$a = b + x$	Definition of >
	$b > c$	Given
	$b = c + y$	Definition of >
	$a = (c + y) + x$	Substitution for "b" in step 2
	$a = c + (y + x)$	A.P.A.
	$y + x \in N$	Closure property of addition
	$a > c$	Definition of >

PROPERTIES OF THE NATURAL NUMBERS UNDER MULTIPLICATION

Multiplication is a binary operation which associates each pair of natural numbers with a unique number. The two natural numbers which comprise the pair are called factors, and the unique number with which they are associated is called the product of the two factors. A "dot" will be used to denote multiplication when two numerals are used as in "$2 \cdot 3$." Juxtaposing two letters or a numeral followed by a letter denotes multiplication as in "ab" (i.e., $a \cdot b$) and "5a" (i.e., $5 \cdot a$). A numeral or letter (or combination of both) to the left of a parenthesized expression also indicates multiplication as in "a (bc)" which means $a \cdot (b \cdot c)$.

1) $ab \in N$ This is called the closure property of multiplication which means that the product of any two natural numbers is also a natural number.

2) $ab = ba$ This is called the commutative property of multiplication which means that the product of two natural numbers is unaffected by the order in which the numbers are combined. We will refer to this property as C.P.M.

3) $a(bc) = (ab)c$ This is called the associative property of multiplication and means that the product of three natural numbers is unaffected by the grouping of the numbers. We will refer to this property as A.P.M.

4) $1a = a$ This is called the multiplication property of 1. The number 1 is called the identity element for multiplication in the set of natural numbers. Combining 1 with any natural number by the operation of multiplication produces that identical number.

A NEW PROPERTY Thus far we have a set of 4 properties relating to addition with natural numbers and a set of 4 properties relating to multiplication with natural numbers. Now we will introduce a property which enables us to combine addition with multiplication as follows:

$a(b + c) = a \cdot b + a \cdot c$ (or simply $ab + ac$)

This property is called the distributive property of multiplication over addition. According to the symmetric property of equality it is also true that $ab + ac = a(b + c)$. We can also assume, for convenience, that $(b + c)a = ba + ca$ and $ba + ca = (b + c)a$. A study of number systems which are beyond the scope of this workshop will reveal the fact that the distributive property governs addition throughout the set of real and complex numbers. The following illustration will show that the familiar multiplication algorithm learned in elementary school is an application of the distributive property.

$$\begin{aligned} 23 \times 32 = (20+3)(30+2) &= 20(30+2) + 3(30+2) \\ &= 20 \cdot 32 + 3 \cdot 32 \\ &= 640 + 96 \\ &= 736 \end{aligned}$$

Throughout the study of elementary mathematics an understanding of the distributive property is an indispensable tool for the mathematics student. Consider the following case in point. A beginning algebra student can simply be told that for any number, n, $n + n = 2n$ or he can be helped to arrive at this conslusion by applying the distributive property in conjunction with some other accepted properties as follows:

$a = 1a$	Identity element for multiplication
$1a + 1a = (1 + 1)a$	Distributive property
$= 2 \cdot a$	Substitution
$= 2a$	Definition of $a \cdot b$

We will refer to this property as D. P. M. A.

An expression such as $2 \cdot 3 + 4$ is ambiguous unless the order in which the operations are to be done is specified. Asking a group of students to simplify such an expression and seeing the multiplicity of answers which result will dramatize the need for a rule to cover such cases. The rule in question is referred to as the order of operations and reads as follows:

In a series of operations which involve any combination of addition, subtraction, multiplication, and division, all multiplications and divisions are to be done first. See examples shown:

a) $2 \cdot 3 + 4 - 5 = 6 + 4 - 5 = 10 - 5 = 5$

b) $2 + 3 \cdot 4 - 5 = 2 + 12 - 5 = 14 - 5 = 9$

c) $2 \cdot 3 + 4 \cdot 5 = 6 + 20 = 26$

Once the rule has been stated, parenthesis are needed only to change the accepted order of operations, e.g., $(3 + 4) \cdot 2$ to show that addition comes first.
It should be noted that when multiplication and division come in immediate succession, the order in which they are performed does not affect the result.

$2 \cdot (6 \div 3) = (2 \cdot 6) \div 3$

$12 \div (3 \cdot 2) = (12 \div 3) \cdot 2$

The same reasoning applies to addition and subtraction.
WARNING: This does not mean that division is associative, nor does it mean that subtraction is associative.

$a \div (b \div c) \neq (a \div b) \div c$

$a - (b - c) \neq (a - b) - c$

THE MULTIPLICATION PROPERTY OF EQUALITY If $a = b$, then $ac = bc$. *

*The converse of this property will be discussed further along in this section.

THE MULTIPLICATION PROPERTY OF ZERO $n \cdot 0 = 0$

Proof:	$0 + 0 = 0$	Identity element for addition
	$n(0 + 0) = n \cdot 0$	Multiplication property of " = "
	$n \cdot 0 + n \cdot 0 = n \cdot 0$	D. P. M. A.
	$n \cdot 0 = 0$	Definition of identity element for addition.

Remember that according to the definition of the identity element for addition if $a + x = a$, then $x = 0$. In the above example $a = a \cdot 0$ and $x = n \cdot 0$; and since their sum is 0, $n \cdot 0$ must be 0.

DIVISION IN THE SET OF NATURAL NUMBERS We will define division in terms of multiplication as follows:

If $c \div b = a$, then $c = a \cdot b$ ("$\div$" is read "divided by").
If "a" is replaced in the second sentence above by its equivalent "$c \div b$", the result is $c = c \div b \cdot b$. This leads to the following definition: "$c \div b$ is the number which when multiplied by b produces c."

DIVISION PROPERTIES OF ZERO

Case I: $0 \div n$ where $n \neq 0$

If $0 \div n = x$ then $x \cdot n = 0$	Definition of division
Either $n = 0$ or $x = 0$	Multiplication property of zero
$n \neq 0$	Given

Therefore $x = 0$

Zero divided by any non-zero natural number equals zero.

Case II: $n \div 0$ where $n \neq 0$

If $n \div 0 = x$ then $x \cdot 0 = n$	Definition of division
$x \cdot 0 = 0$	Multiplication property of zero.
$n \neq 0$	Given

In this case "$n \div 0$" is a meaningless symbol because no natural number times zero will produce a non-zero product.

Case III: $0 \div 0$

If $0 \div 0 = x$ then $x \cdot 0 = 0$ — Definition of division

But Any $x \cdot 0 = 0$ — Multiplication property of zero.

In this case "$0 \div 0$" is a meaningless symbol because there is no unique number which can be associated with the number pair (0, 0) under the operation of division.

SUMMARY: Division by zero is undefined in the set of natural numbers.

MORE ABOUT THE MULTIPLICATION PROPERTY OF EQUALITY:

It was stated earlier that if $a = b$, then $ac = bc$. Now let us examine the converse of this statement, namely, if $ac = bc$, then $a = b$. Consider the statement $2 \cdot 0 = 3 \cdot 0$ which is certainly true according to the properties which have been accepted so far. However, if we accept, without reservation, the converse of the previously stated multiplication property of equality we would be forced to accept the statement $2 = 3$, which is obviously nonsensical.

SUMMARY: If $a = b$, then $ac = bc$.
If $ac = bc$ and $c \neq 0$, then $a = b$.

SOLUTION SETS OF OPEN SENTENCES IN THE SET OF NATURAL NUMBERS

DEFINITION OF SOLUTION SET The solution set of an open sentence is the set of all members of a specified universal set whose names can be used to make true statements when substituted for the variables in the open sentence.

For any $b \leq a$ there exists an x such that $b + x = a$. With the properties which have been established to this point, the student has sufficient tools to find the solution set of some open sentence of the form $nx + b = c$ in the set of natural numbers. See the following example:

$U = N$ Find the solution set of $2x + 3 = 11$

$2x + 3 = 11$	Given
$2x = 11 - 3$	Definition of Subtraction
$2x = 8$	Substitution
$x = 8 \div 2$	Definition of division
$x = 4$	Substitution

According to one school of thought at this point only the first of two large steps toward finding the solution set has been completed, namely that of eliminating non-possible members of the solution set. The second step consists of testing the possible candidate for membership in the solution set by substituting its name for the variable in the original sentence. If a true statement results for each substitution, the solution set has been found.

$$2 \cdot 4 + 3 = 11$$
$$8 + 3 = 11$$

Since 4 was the only candidate, we can indicate the solution set as follows: The solution set is $\{4\}$.

- continued

This two step procedure corresponds to the logical process of examining a true statement and its converse.

Statement: If $2x + 3 = 11$, then $x = 4$.

Converse: If $x = 4$, then $2x + 3 = 11$.

According to a second school of thought on this matter the second step is unnecessary if all sentences subsequent to the first sentence are obtained by means which ensure that these sentences are equivalent to the original sentence.

It must be noted that the student will be unable, at this point, to find solution sets of certain variations of $nx + b = c$ if he is restricted to only those properties which have been established. Consider the following example:

$U = N$ Find the solution set of $5x = 2x + 9$

$5x = 2x + 9$	Given
$5x - 2x = 9$	Definition of subtraction

Now at this point the student knows intuitively that $5x - 2x = 3x$; however, since distributivity of multiplication over subtraction has not been defined, the above sentence cannot be simplified further, using only the available properties. In the next chapter we will see that the operation of subtraction can be dispensed with so that a situation such as the one above is no obstacle.

EXERCISES:

1) In each of the following exercises you may use any of the properties which have been discussed.
   a) Prove that subtraction is not commutative
   b) Prove that $2n + 3n = 5n$

2) Use the distributive property to write each indicated sum as a product.
   a) $12 + 15$; b) $a + ab$; c) $9ab + 6b$; d) $ab + ac + bd + cd$

3) 3) Use the distributive property to write each indicated product as a sum.
   $2(a + 3)$; b) $a(2b + 3c)$; c) $5a(2b + 3c)$; d) $b(2c + 1)$

4) Why can't the distributive property be used to write the sum $15 + 16$ as an indicated product?

5) Give numerical examples to show that
   a) The set of natural numbers is not closed for subtraction
   b) Subtraction is not associative
   c) Division is not commutative
   d) Division is not associative
   e) The set of natural numbers is not closed for division

6) Prove that $a + c > b + c \longrightarrow a > b$

ANSWERS TO EXERCISES:

1a) To prove that $a - b = b - a$ is not a true sentence; we need to find some replacements for a and b, which will make it false. Since $3 - 2 = 2 - 3$ is a false sentence, the generalization $a - b = b - a$ is not a true sentence for all a N and b N.

1b)
$$\begin{aligned} 2n + 3n &= (2 + 3)n && \text{Distributive property} \\ &= 5 \cdot n && \text{Substitution} \\ &= 5n && \text{Symbolism for product} \end{aligned}$$

2) a) $3(4 + 5)$; b) $a(1 + b)$; c) $3b(3a + 2)$; d) $a(b + c) + d(b + c)$
$= (a + d)(b + c)$

3) a) $2a + 6$; b) $2ab + 3ac$; c) $10ab + 15ac$; d) $2bc + b$

4) Because 15 and 16 have no common divisor other than 1

5) a) $4 - 5$; b)
$$\begin{aligned} 9 - (5 - 3) &\neq (9 - 5) - 3 \\ 9 - 2 &\neq 4 - 3 \\ 7 &\neq 1 \end{aligned}$$

c)
$$\begin{aligned} 4 \div 2 &\neq 2 \div 4 \\ 2 &\neq 2 \div 4 \end{aligned}$$
; d)
$$\begin{aligned} 12 \div (6 \div 2) &\neq (12 \div 6) \div 2 \\ 12 \div 3 &\neq 2 \div 2 \\ 4 &\neq 1 \end{aligned}$$

e) $2 \div 3$

6) If $a + c > b + c$ then
$$\begin{aligned} a + c &= (b + c) + x && \text{Definition of} > \\ a + c &= b + (c + x) && \text{A. P. A.} \\ a + c &= b + (x + c) && \text{C. P. A.} \\ a + c &= (b + x) + c && \text{A. P. A.} \\ a &= b + x && \text{Addition axiom of} = \\ a &> b && \text{Definition of} > \end{aligned}$$

UNIT III

THE INTEGERS

PRELIMINARY DISCUSSION It was stated in the section on natural numbers that there is a non-empty solution set in the set of natural numbers for every open sentence $b + x = a$ where $b \leq a$. Also, there is an infinitude of such open sentences where $b > a$; e.g., $2 + x = 1$. There are no solutions for such sentences in the set of natural numbers, so a set which will provide solutions had to be invented. This new set of numbers is called the set of integers, and we are going to show several different methods which can be used to develop the system of integers with junior high school classes.

THE SET OF INTEGERS We can use the idea of the Farenheit thermometer to develop the set of integers. The numbers represented on a thermometer constitute a "number line" on which the number zero is arbitrarily chosen to separate the line into three sets: zero, numbers above zero, and numbers below zero.

If we designate the numbers above zero "positive" and the numbers below zero "negative," we can call the union of the three sets, the set of integers.

$\{\ldots{}^-4, {}^-3, {}^-2, {}^-1\} \cup \{0\} \cup \{^+1, {}^+2, {}^+3, {}^+4 \ \ldots\} = \{\text{Integers}\}$

Integers are sometimes called directed numbers. Each integer has magnitude and, with the exception of zero, indicates one of two directions, positive or negative. The order of magnitude is from negative through zero to positive. Thus $^-1$ is one less than zero, $^-2$ is one less than $^-1$, etc. Because of their behavior, the non-negative integers make up a set which is said to be isomorphic to the set of natural numbers. In practice the integer $^+1$, for example, is written as "1"; however, it must be emphasized that $^+1$ and 1 belong to two different sets. We don't say "positive one" when we count, nor does the word "one" indicate direction.

SUMMARY: $\{\text{Negative Integers}\} \cup \{\text{Zero Integer}\} \cup \{\text{Positive Integers}\} = \{\text{Integers}\}$

THE SYSTEM OF INTEGERS Depending upon the caliber of the class, operations in the set of integers can be developed by inductive methods which make use of demonstrations and appeal to the intuition, or an axiomatic approach can be used.

THE NUMBER LINE APPROACH TO THE ADDITION OF INTEGERS

$$\ldots \quad {}^-4 \quad {}^-3 \quad {}^-2 \quad {}^-1 \quad 0 \quad {}^+1 \quad {}^+2 \quad {}^+3 \quad {}^+4 \quad \ldots$$

Rules: Moving k units to the right on the "integer line" denotes addition of the integer "positive k".

Moving k units to the left on the integer line denotes addition of the integer "negative k".

By applying the above two rules integer sums can be easily determined. A more detailed discussion of this method as used to illustrate subtraction, multiplication and division can be found in an article by L. H. Coon in the Arithmetic Teacher, Vol. 13, Number 3, pages 213-217.

THE NOMOGRAPH APPROACH TO ADDITION AND SUBTRACTION

This method relies for its effectiveness upon the students understanding of the definition of addition-subtraction for any numbers, namely

If $a + b = c$ then $a = c - b$ and $b = c - a$

- continued

The nomograph consists of three number scales evenly spaced and numbered as in the illustration below:

a	c	b
$^{+}4$.	$^{+}8$.	. $^{+}4$
	$^{+}7$.	
$^{+}3$.	$^{+}6$.	. $^{+}3$
	$^{+}5$.	
$^{+}2$.	$^{+}4$.	. $^{+}2$
	$^{+}3$.	
$^{+}1$.	$^{+}2$.	. $^{+}1$
	$^{+}1$.	
0 .	0 .	. 0
	$^{-}1$.	
$^{-}1$.	$^{-}2$.	. $^{-}1$
	$^{-}3$.	
$^{-}2$.	$^{-}4$.	. $^{-}2$
	$^{-}5$.	
$^{-}3$.	$^{-}6$.	. $^{-}3$
	$^{-}7$.	
$^{-}4$.	$^{-}8$.	. $^{-}4$

The sum a + b is found by connecting the appropriate numerals in the "a" and "b" scales with a string and reading the result where the string intersects the "c" scale. In the same way, the difference c - b can be found as a projection on the "a" scale and c - a can be found as a projection on the "b" scale. The disadvantage of the nomograph is that logarithmic spacing of the numerals is necessary in order to illustrate multiplication and division.

<u>MORE METHODS FOR DERIVING INTEGER PROPERTIES USING NATURAL NUMBER PROPERTIES AND SPECIFIC NUMERICAL EXAMPLES</u> From the above heading, it can be seen that this is an inductive method since mathematical generalizations of an affirmative nature cannot be reached by the observation of specific cases*. Of course one counter-example is sufficient to reach a generalization of a negative nature; e.g., 2 - 3 is sufficient reason to make the statement that the set of natural numbers is not closed under the operation of subtraction.

* In number theory a form of logical reasoning called <u>mathematical induction</u> is used.

In using this method we will make the following definitions:

<u>Definition 1</u>: For each integer, <u>a</u>, there is another integer called the <u>negative of a</u> or (the <u>opposite of a</u>);

e.g., $^{-}1$ is the opposite of $^{+}1$, $^{+}2$ is the opposite of $^{-}2$ etc.

<u>Definition 2</u>: For any integer, <u>a</u>, a + $^{-}$a = 0. $^{-}$a is called the <u>additive inverse</u> of <u>a</u>;

e.g., $^{-}1$ + $^{+}1$ = 0, $^{+}2$ + $^{-}2$ = 0 etc.

Many texts distinguish typographically between the operation of addition and the prefix "positive" by positioning the symbol "+" as is done in the example which accompanies definition (2). In such texts the same typographical device is used for the operation of subtraction and the prefix "negative" with respect to the symbol "-".

Example 1: Find the simplest name for $5 + {}^{-}2$

Let $5 + {}^{-}2 = x$	
$5 + ({}^{-}2 + 2) = x + 2$	Addition axiom of = and A. P. A.
$5 + 0 = x + 2$	Substitution according to the definition of additive inverse
$5 = x + 2$	Identity element for addition
$5 - 2 = x$	Definition of subtraction
$3 = x$	Substitution
$x = 3$	Symmetric property of =
$5 + {}^{-}2 = 3$	Transitive property of =
$3 = 5 - 2$	Substitution
$5 + {}^{-}2 = 5 - 2$	Transitive property of =

It is to be hoped that the students will, through the use of hints and suggestions, be encouraged to derive the above result, as well as those to follow, for themselves. In this way the derived properties will be accepted as results which follow in a perfectly logical manner from the original assumptions and not as rules which are arbitrarily handed down for use.

AN ALTERNATE PROCEDURE FOR EXAMPLE 1:

$5 + {}^{-}2 = (3 + 2) + {}^{-}2$	Of course this step simply makes use of the substitution principle. Since it provides the key to this particular procedure ample time should be allowed for the students to think of this step themselves.
$= 3 + (2 + {}^{-}2)$	A. P. A.
$= 3 + 0$	Substitution
$= 3$	Identity element for addition
$3 = 5 - 2$	Substitution
$5 + {}^{-}2 = 5 - 2$	Transitive property of =

This and other examples will lead to the assumption "for a, b>0 and a > b $a + {}^{-}b = a - b$".

Example 2: Find the simplest name for ${}^{-}3 + {}^{-}2$

Let ${}^{-}3 + {}^{-}2 = x$	
${}^{-}3 + ({}^{-}2 + 2) = x + 2$	Addition axiom of = and A. P. A.
${}^{-}3 + 0 = x + 2$	Additive inverse
${}^{-}3 = x + 2$	Identity element for addition
${}^{-}3 + 3 = x + (2+3)$	Addition axiom of = and A. P. A.
$0 = x + 5$	Substitution
$0 + {}^{-}5 = x + (5 + {}^{-}5)$	Addition axiom of = and A. P. A.
${}^{-}5 = x + 0$	Substitution
${}^{-}5 = x$	Identity element for addition
$x = {}^{-}5$	Symmetric property of =
${}^{-}3 + {}^{-}2 = {}^{-}5$	Transitive property of =

By definition, ${}^{-}5$ means ${}^{-}(5)$; i.e., ${}^{-}(4 + 1)$, ${}^{-}(3 + 2)$ etc. This will lead to the assumption "for $a, b > 0$, ${}^{-}a + {}^{-}b = {}^{-}(a + b)$".

Example 3: Find the simplest name for ${}^{-}5 + 2$

${}^{-}5 + 2 = ({}^{-}3 + {}^{-}2) + 2$	Substitution according to the previous example
$= {}^{-}3 + ({}^{-}2 + 2)$	A. P. A.
$= {}^{-}3 + 0$	Additive inverse
$= {}^{-}3$	Identity element for addition
$= {}^{-}(5 - 2)$	

This leads to the assumption "for $a,b > 0$ and $a > b$, ${}^{-}a + b = {}^{-}(b - a)$".

Example 4: Find the simplest name for $2 - 5$

Let $2 - 5 = x$

$2 = x + 5$	Definition of subtraction
$2 + {}^{-}2 = x + (5 + {}^{-}2)$	Addition axiom of = and A. P. A.
$0 = x + 3$	Additive inverse and substitution according to result in Example 1.
$0 + {}^{-}3 = x + (3 + {}^{-}3)$	Addition axiom of =
${}^{-}3 = x + 0$	Identity element for addition and additive inverse
${}^{-}3 = x$	Identity element for addition
$x = {}^{-}3$	Symmetric property of =
$2 - 5 = {}^{-}3$	Transitive property of =
${}^{-}3 = 2 + {}^{-}5$	Result in Example 3
$2 - 5 = 2 + {}^{-}5$	Transitive property of =

It will be left to the reader as an exercise to verify that ${}^{-}2 - 5 = {}^{-}2 + {}^{-}5$. This result combined with the results from Examples 1, 3, and 4 can be stated as the single assumption.

For any integers a and b, $a + {}^{-}b = a - b$

In the next section it will be seen that this result can be proved for all cases in one series of steps.

Example 5: Find the simplest name for $2 - {}^{-}5$

Let $2 - {}^{-}5 = x$

$2 = x + {}^{-}5$	Definition of subtraction
$2 = x - 5$	Result of previous examples
$2 + 5 = x$	Definition of subtraction
$7 = x$	Substitution
$x = 7$	Symmetric property of =
$2 - {}^{-}5 = 7$	Transitive property of =
$7 = 2 + 5$	Substitution
$2 - {}^{-}5 = 2 + 5$	Transitive property of =

It will be left to the reader as an exercise to verify that ${}^{-}2 - {}^{-}5 = {}^{-}2 + 5$.

These and other examples lead to the assumption "$a - {}^{-}b = a + b$ for any integers a and b".

Example 6: Find the simplest name for $2 \cdot {}^-3$

${}^-3 + 3 = 0$	Definition of additive inverse
$2({}^-3 + 3) = 2 \cdot 0$	Multiplication axiom of =
$2 \cdot {}^-3 + 2 \cdot 3 = 0$	D. P. M. A. and multiplication property of 0
$2 \cdot {}^-3 + 6 = 0$	Substitution
$2 \cdot {}^-3 + (6 + {}^-6) = 0 + {}^-6$	Addition axiom of =
$2 \cdot {}^-3 + 0 = {}^-6$	Substitution
$2 \cdot {}^-3 = {}^-6$	Identity element for addition
$= {}^-(2 \cdot 3)$	

This leads to the assumption "for $a, b > 0$, $a \cdot {}^-b = {}^-(ab)$".

Example 7: Find the simplest name for ${}^-3 \cdot {}^-2$

${}^-2 + 2 = 0$	Definition of additive inverse
${}^-3({}^-2 + 2) = {}^-3 \cdot 0$	Multiplication axiom of =
${}^-3 \cdot {}^-2 + {}^-3 \cdot 2 = 0$	D. P. M. A. and multiplication property of 0
${}^-3 \cdot {}^-2 + {}^-6 = 0$	Substitution and C. P. M.
${}^-3 \cdot {}^-2 + {}^-6 + 6 = 0 + 6$	Addition axiom of =
${}^-3 \cdot {}^-2 + 0 = 6$	Substitution
${}^-3 \cdot {}^-2 = 6$	Identity element for addition
$= 3 \cdot 2$	

This leads to the assumption "for $a, b > 0$, ${}^-a \cdot {}^-b = ab$".

DIVISION IN THE SET OF INTEGERS The rules for division in the set of integers can be derived readily by using the definition of division and the results obtained in Examples 6 and 7.

Example 8: Find the simplest name for ${}^-6 \div 2$

If ${}^-6 \div 2 = x$, then $2x = {}^-6$ Definition of division

Therefore, x is that number which when multiplied by 2 produces a product of ${}^-6$

From Example 6, x must be ${}^-3$.

Example 9: Find the simplest name for $6 \div {}^-2$

If $6 \div {}^-2 = x$, then ${}^-2x = 6$. Definition of division

Therefore, x is that number which when multiplied by ${}^-2$ produces a product of 6.

From Example 7, x must be ${}^-3$.

ANOTHER PROPERTY OF > :

In the set of natural numbers this is true: $c \neq 0$ and $a > b$, if and only if $ac > bc$.

Proof: If $a > b$, then $a = b + x$ — Definition of >

$a = b + x$	Definition of >
$c(a) = c(b + x)$	Multiplication axiom of =
$ac = bc + xc$	D. P. M. A.
$ac > bc$	Definition of >

The converse of the above can be proved by merely reversing the steps in a proof to be certain that the steps are indeed valid when reversed.

Now we must examine this property in the set of integers.

$$3 > 2 \text{ but } {}^{-}5(3) < {}^{-}5(2)$$
$$^{-}1 > {}^{-}2 \text{ but } {}^{-}3({}^{-}1) < {}^{-}3({}^{-}2)$$

As a result of the above examples we must make the following modification when stating this property with respect to the set of integers:

$$a > b \text{ and } c > 0, \text{ if and only if, } ac > bc$$

The property still holds but the restriction on c is enlarged from non-zero to positive.

PROOFS OF SOME INTEGER THEOREMS

In the following proofs we will make use of the properties which apply in the set of natural numbers plus the definition of the additive inverse.

THEOREM I: For any integers a and b, $a + {}^{-}b = a - b$

Let $a + {}^{-}b = x$

$a + ({}^{-}b + b) = x + b$	Addition axiom of = and A. P. A.
$a + 0 = x + b$	Additive inverse
$a = x + b$	Identity element for addition
$a - b = x$	Definition of subtraction
$x = a - b$	Symmetric property of =
$a + {}^{-}b = a - b$	Transitive property of =

THEOREM II: ${}^{-}a + {}^{-}b = {}^{-}(a + b)$

Let ${}^{-}a + {}^{-}b = x$

${}^{-}a + ({}^{-}b + b) = x + b$	Addition axiom of = and A. P. A.
${}^{-}a + 0 = x + b$	Additive inverse
${}^{-}a = x + b$	Identity element for addition
${}^{-}a + a = x + (b + a)$	Addition axiom of =
$0 = x + (a + b)$	Additive inverse and C. P. A.
$0 + {}^{-}(a + b) = x + (a + b) + {}^{-}(a + b)$	Addition axiom of =
${}^{-}(a + b) = x$	Identity element for addition and Additive inverse
$x = {}^{-}(a + b)$	Symmetric property of =
${}^{-}a + {}^{-}b = {}^{-}(a + b)$	

THEOREM III: $a - ({}^{-}b) = a + b$

Let $a - ({}^{-}b)$	$= x$	
a	$= x + ({}^{-}b)$	Definition of subtraction
a	$= x - b$	Theorem I
$a + b$	$= x$	Definition of subtraction
x	$= a + b$	Symmetric property of =
$a - ({}^{-}b)$	$= a + b$	Transitive property of =

THEOREM IV: ${}^{-}({}^{-}a) = a$

${}^{-}({}^{-}a) + {}^{-}a = 0$	Definition of additive inverse
$0 = a + {}^{-}a$	Definition of additive inverse
${}^{-}({}^{-}a) + {}^{-}a = a + {}^{-}a$	Transitive property of =
${}^{-}({}^{-}a) = a$	Addition axiom of =

THEOREM V: $a({}^{-}b) = {}^{-}(ab)$

$a({}^{-}b + b) = a(0)$	Substitution
$a({}^{-}b) + ab = 0$	D. P. M. A. and multiplication property of 0
$a({}^{-}b) + \{ab + {}^{-}(ab)\} = 0 + {}^{-}(ab)$	Addition axiom of = and A. P. A.
$a({}^{-}b) + 0 = {}^{-}(ab)$	Additive inverse and identity element for addition
$a({}^{-}b) = {}^{-}(ab)$	

THEOREM VI: ${}^{-}a({}^{-}b) = ab$

${}^{-}a({}^{-}b + b) = {}^{-}a(0)$	Substitution
${}^{-}a({}^{-}b) + {}^{-}(ab) = 0$	D. P. M. A., theorem I and multiplication property of)
${}^{-}a({}^{-}b) + \{{}^{-}(ab) + ab\} = 0 + ab$	Addition axiom of = and A. P. A.
${}^{-}a({}^{-}b) + 0 = ab$	Additive inverse and identity element for addition
${}^{-}a({}^{-}b) = ab$	Identity element for addition

EXERCISES

1) Find the simplest name for each of the following numerals by making use of the previously proved theorems:

a) ${}^{-}7 + 9$; b) ${}^{-}13 + 4$; c) ${}^{-}9 + {}^{-}11$; d) $8 + {}^{-}13$; e) $0 + {}^{-}17$

f) $6 - ({}^{-}7)$; g) ${}^{-}9 - 9$; h) $0 - ({}^{-}3)$; i) ${}^{-}8 - ({}^{-}9)$; j) ${}^{-}1 - ({}^{-}1)$

k) ${}^{-}16 \div 4$; l) $72 \div {}^{-}9$

2) Prove that $a - (b + c) = a - b - c$

3) Prove that $a - (b - c) = a - b + c$

ANSWERS TO EXERCISES:

1) a) $^{+}2$; b) $^{-}9$; c) $^{-}20$; d) $^{-}5$; e) $^{-}17$; f) $^{+}13$; g) $^{-}18$; h) $^{+}3$
i) $^{+}1$; j) 0 ; k) -4 ; l) -8

2) Let $a - (b + c) = x$

a	$= x + (c + b)$	Definition of subtraction and C. P. A.
a	$= (x + c) + b$	A. P. A.
$a - b$	$= x + c$	Definition of subtraction
$a - b - c$	$= x$	Definition of subtraction
x	$= a - b - c$	Symmetric property of =
$a - (b + c)$	$= a - b - c$	Transitive property of =

3) Let $a - (b - c) = x$

a	$= x + (b - c)$	Definition of subtraction
a	$= x + (b + {}^{-}c)$	Theorem I
a	$= (x + b) + {}^{-}c$	A.P.A.
$a + c$	$= (x + b) + ({}^{-}c + c)$	Addition axiom of equality
$a + c$	$= x + b + 0$	Additive inverse
$a + c$	$= x + b$	Identity element for addition
$(a + c) + {}^{-}b$	$= x + (b + {}^{-}b)$	Addition axiom of = and A. P. A.
$(a + c) + {}^{-}b$	$= x + 0$	Additive inverse
$(a + c) + {}^{-}b$	$= x$	Identity element for addition
$a + (c + {}^{-}b)$	$= x$	A. P. A.
$a + ({}^{-}b + c)$	$= x$	C. P. A.
$(a + {}^{-}b) + c$	$= x$	A. P. A.
$a - b + c$	$= x$	Theorem I
x	$= a - b + c$	Symmetric property of =
$a - (b - c)$	$= a - b + c$	Transitive property of =

AN ALTERNATE APPROACH TO PROOFS OF INTEGER THEOREMS

This is quite a sophisticated approach in which all integer theorems are proved by defining an integer in terms of natural numbers and using the properties which govern operations in the set of natural numbers plus the definitions of equality, addition and multiplication in the set of integers. We will give a few proofs using this approach. For a detailed discussion of this approach the reader can consult "Integers", a pamphlet in the "Thinking in Mathematics" series published by D. C. Heath and Company, or "Introduction to Mathematical Thinking" by Waissmann, a Harper book.

Definition I: An integer is the difference of two natural numbers. The difference $a - b$ will be indicated as the ordered pair (a, b). Each integer is an equivalence class as illustrated by the following examples:

$$^{+}2 = \{ (2, 0), (3, 1), (4, 2) \ldots (n + 2, n) \}$$

$$^{-}3 = \{ (0, 3), (1, 4), (2, 5) \ldots (n, n + 3) \}$$

In particular $0 = \{(0, 0), (1, 1), (2, 2) \ldots (n, n) \}$
and $^{+}1 = \{ (1, 0), (2, 1), (3, 2) \ldots (n + 1, n) \}$

Definition II: Equality of two integers. Based upon the discussion of an integer as an equivalence class it seems reasonable to make the following definition of equality:

$$(a, b) = (c, d) \Longleftrightarrow (a + d) = (b + c)$$

Definition III: Addition. The assumption that the set of non-negative integers is isomorphic with the set of natural numbers combined with the definitions above seems to make the following reasonable definition for addition:

$$(a, b) + (c, d) = (a + c,\ b + d)$$

THEOREM I: CLOSURE PROPERTY OF ADDITION

Let $I = \{\text{Integers}\}$. $(a, b) + (c, d) \in I$

$a + c \in$ Natural Numbers (N)	Closure property of N for +
$b + d \in N$	Closure property of N for +
$(a, b) + (c, d) = (a + c,\ b + d)$	Definition of + in I

Since $(a + c,\ b + d)$ is the difference of two natural numbers, by definition it must be an integer.

THEOREM II: COMMUTATIVE PROPERTY OF ADDITION

$(a, b) + (c, d) = (c, d) + (a, b)$

$(a, b) + (c, d) = (a + c,\ b + d)$	Definition of + in I
$= (c + a,\ d + b)$	C. P. A. in N
$= (c, d) + (a, b)$	Definition of + in I

THEOREM III: ASSOCIATIVE PROPERTY OF ADDITION

This should be done by the interested reader as an exercise using theorem II as a guide.

THEOREM IV: IDENTITY ELEMENT FOR ADDITION

If there is an identity element for addition in I, it can be found by posing the following task:

Find $\{ (x, y) : (a, b) + (x, y) = (a, b) \}$

$(a, b) + (x, y) = (a + x,\ b + y)$	Definition of + in I
If $(a + x,\ b + y) = (a, b)$ then $a + x = a$	Definition of = in I
If $a + x = a$ then $x = 0$	Identity element for addition in N

By the same reasoning $y = 0$

Therefore, the identity element for addition in $I = (0,0)$

UNIT IV

THE RATIONAL NUMBER SYSTEM

INTRODUCTION

It must be properly noted that mathematics has always been invented as a result of two motives: (1) Academic interest, and (2) practical need. Fractions - their innovation and use - were the result of the latter. In the quest to refine and improve ways of measuring, man soon realized that he needed more than natural numbers; he needed parts of numbers.

That this need was felt very early is evident in the famous "Rhind Papyrus", an Egyptian Document some 3500 years old. In this manuscript, the use of fractions is pointedly referred to. However, it is significant that the Egyptians did not develop their magnificent discovery. This lack of enthusiasm is evident in much of mathematics history. It seems as though the development of mathematics was taken just as far as practical need required and no further.

In any case it seems as though a rigorous development of the kind of number, represented by the fraction, was to be left to the mathematician.

THE INVENTION OF THE RATIONAL NUMBER

The invention of the system of integers came about in an effort to solve the open sentence:

> $a + x = b$, where a and b are natural numbers.

Now consider the open sentence:

> $a \cdot x = b$, where a and b are members of the set of integers with $a \neq 0$. To solve for x in this open sentence leads to some interesting results. It may seem as though an integral solution is possible, thinking in terms of b as some multiple of a, i. e., $a = 3$, $b = 9$;

$3x = 9$

$3x = 3 \cdot 3$ (renaming 9)

$x = 3$ (cancellation)

Thus the solution is the integer 3.

After such an example, the premature statement may be made that no new number is needed to find the solution for such sentences, i. e., the set of integers is sufficient. However, when a and b are prime or relatively prime (b is not a multiple of a), a different situation occurs. Consider the following example:

$3 \cdot x = 11$

Using the set of integers as the universal set, there is no solution for the variable x such that a true statement will result.

To solve this dilemma, it is necessary to invent a new number system which will generate a solution to all open sentences of the form:

$$a + x = b, \text{ where } a, b \in I \text{ and } a \neq 0$$

We shall define this new number which will be called a Rational Number in the following way:

DEFINITION OF A RATIONAL NUMBER If a and b $\in$ I and a $\neq$ 0, then there exists some x, called a Rational Number, such that

$$x = \frac{b}{a}$$

In effect, by defining the Rational Number in this way, we could at the same time define the operation of division, with x being the quotient of two integers.

SYMBOL FOR A RATIONAL NUMBER A Rational Number is a number and as such cannot have a numerator or denominator. However, the symbol for a Rational Number which is called a Fraction, can and does have a numerator and denominator.

INTEGERS AS A SUBSET OF THE RATIONAL NUMBERS The set of integers constitute a subset of the set of rational numbers since every integer expressed is the ratio of two integers.

Example:

INTEGER	RATIONAL NUMBER
+3	$\frac{+3}{+1}$
(- 7)	$\frac{-7}{1}$
0	$\frac{0}{+1}$

Exercises:

(1) Why are the following considered names for rational numbers?

(a) $\frac{3}{5}$

(b) $\frac{8}{6}$

(c) $\frac{10}{12}$

(2) If x, y and z $\in$ I, state whether the following are rational numbers and justify your answer.

(a) $\frac{x}{3}$

(b) $\frac{x + y}{7}$

(c) $\frac{6x + 5z}{9}$

- continued

(3) Using the information given in problem (2),

$\frac{6x}{z}$ would not necessarily be a rational number.

Can you reason why this is so?

Exercises (Answers)

(1) All the fractions represent rational numbers since each fraction is in the form $\frac{a}{b}$ with both a and b, integers and $b \neq 0$.

(2) (a) $\frac{x}{3}$ is a rational number because it is in the proper form and $3 \neq 0$

(b) $\frac{x+y}{7}$ is a rational number because of the form. Also, $x + y$ is an integer because of closure.

(c) $6x$ is an integer because of closure; $5z$ is an integer because of closure. $6x + 5z$ is an integer because of closure. Therefore, $\frac{6x + 5z}{9}$ represents a rational number.

(d) 0 is an integer and z might be 0. The denominator of the fraction which represents a rational number may not be equal to 0.

EQUIVALENCE CLASSES IN THE SET OF RATIONAL NUMBERS -

Recall that the principle of substitution uses the idea that a number has many names. Thus, $2 = 1 + 1$. The same idea may be extended to the set of rational numbers. There exists an infinite number of rational numbers which are equivalent to one rational number. This infinite set is called an Equivalence Class.

DEFINITION OF EQUIVALENT RATIONAL NUMBERS Two rational numbers, $\frac{a}{b}$ and $\frac{c}{d}$, where a, b, c, and $d \in I$ and $b \neq 0$, $d \neq 0$ are said to be equivalent if and only if $ad = bc$.

Using this definition, it can be seen that $\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{5}{15}, \ldots$ are rational numbers and that these numbers constitute an infinite set called an equivalence class.

Also, any equivalence class can be represented by one of its elements. Thus $\frac{14}{28}$ names an equivalence class the basic member of which is $\frac{1}{2}$.

By a basic member, or basic fraction, we mean that member of the equivalence class whose numerator and denominator have no common factor other than +1 or -1.

- continued

Exercises:

(1) Name five members of the equivalence classes named by the following rational numbers.

(a) $\frac{1}{8}$ (b) $\frac{8}{7}$ (c) $\frac{4}{3}$ (d) $\frac{7}{9}$

Exercises (Answers)

(1) (a) $\frac{1}{8}, \frac{2}{16}, \frac{3}{24}, \frac{4}{32}, \frac{5}{40}, \ldots$

(b) $\frac{8}{7}, \frac{16}{14}, \frac{24}{21}, \frac{32}{28}, \frac{40}{35}, \ldots$

(c) $\frac{4}{3}, \frac{8}{6}, \frac{12}{9}, \frac{16}{12}, \frac{20}{15}, \ldots$

(d) $\frac{7}{9}, \frac{14}{18}, \frac{21}{27}, \frac{28}{36}, \frac{35}{45}, \ldots$

RELATIONS WITH RATIONAL NUMBERS Relations with rational numbers can be as clearly defined as relations in the systems already discussed.

EQUALS RELATION For any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$

$\frac{a}{b} = \frac{c}{d}$, if and only if ($\Longleftrightarrow$), $ad = bc$

Example: $\frac{3}{4} = \frac{6}{8}$ because $3 \cdot 8 = 4 \cdot 6$. Also because this is an "if and only if" statement it can be said that if

$3 \cdot 8 = 4 \cdot 6$ then $\frac{3}{4} = \frac{6}{8}$

INEQUALITIES AND RATIONAL NUMBERS The familiar "less than" and "greater than" relations can also be applied to rational numbers.

"LESS THAN" RELATION

For any two rational numbers $\frac{a}{b} < \frac{c}{d} \Longleftrightarrow ad < bc$

"GREATER THAN" RELATION

For any two rational numbers $\frac{a}{b} > \frac{c}{d} \Longleftrightarrow ad > bc$

Exercises:

(1) Establish the relation of the following rational numbers by using the principles just discussed.

(a) $\frac{6}{9}, \frac{8}{11}$, (b) $\frac{x}{y}, \frac{2x}{2y}$ (c) $\frac{7}{2}, \frac{8}{6}$

(2) Arrange the following rational numbers from lowest to highest.

$\frac{5}{6}, \frac{7}{8}, \frac{4}{9}, \frac{19}{42}, \frac{6}{12}, \frac{8}{9}$

Exercises (Answers)

(1) (a) $\frac{6}{9} < \frac{8}{11} \Longleftrightarrow 6 \cdot 11 < 9 \cdot 8$

(b) $\frac{x}{y} = \frac{2x}{2y} \Longleftrightarrow x(2y) = y(2x), \quad y \neq 0$

(c) $\frac{7}{2} > \frac{8}{6} \Longleftrightarrow 7 \cdot 6 > 2 \cdot 8$

(2) $\frac{4}{9} < \frac{19}{42} < \frac{6}{12} < \frac{5}{6} < \frac{7}{8} < \frac{8}{9}$

PROPERTIES OF EQUALITY

The properties of equality which have already been defined will be re-stated here. In order for the system of rational numbers to have the equivalence relation these properties must be true for rational numbers.

(1) Reflexive Property For all rational numbers,

$$\frac{a}{b} = \frac{a}{b}$$

(2) Symmetric Property For all rational numbers,

if $\frac{a}{b} = \frac{c}{d}$, then $\frac{c}{d} = \frac{a}{b}$

(3) Transitive Property For all rational numbers,

if $\frac{a}{b} = \frac{c}{d}$ and $\frac{c}{d} = \frac{e}{f}$, then $\frac{a}{b} = \frac{e}{f}$

(4) Additive Property For all rational numbers,

if $\frac{a}{b} = \frac{e}{f}$ then $\frac{a}{b} + \frac{c}{d} = \frac{e}{f} + \frac{c}{d}$

(5) Multiplicative Property For all rational numbers,

if $\frac{a}{b} = \frac{e}{f}$ then, $\frac{a}{b} \cdot \frac{c}{d} = \frac{e}{f} \cdot \frac{c}{d}$

OPERATIONS USING RATIONAL NUMBERS

RATIONAL NUMBERS A basic idea in regards to fractions which will be useful in several instances is the following true sentence.

$$\frac{a}{b} = a \cdot \frac{1}{b};\ a \in I,\ b \in I, \text{ and } b \neq 0.$$

ADDITION OF RATIONAL NUMBERS For any two rational numbers, the symbols for which have equal denominators,

$$\frac{a}{c}, \frac{b}{c};\ \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

The above definition can be verified by means of the distributive property.

$$\frac{a}{c} + \frac{b}{c} = a \cdot \frac{1}{c} + b \cdot \frac{1}{c} = (a+b) \cdot \frac{1}{c} = \frac{a+b}{c},$$

$$\text{and } \frac{a+b}{c} = (a+b) \cdot \frac{1}{c} = a \cdot \frac{1}{c} + b \cdot \frac{1}{c} = \frac{a}{c} + \frac{b}{c}.$$

However, the distributive property for rational numbers has not been assumed at this point and cannot be used in a proof. If it were, the above definition could be referred to as a theorem rather than a definition.

Consider the addition of the rational numbers $\frac{a}{b} + \frac{c}{d} = ?$

$\frac{a}{b} = \frac{ad}{bd}$ since $a(bd) = b(ad)$ Definition of equal rational numbers.

$\frac{c}{d} = \frac{bc}{bd}$ since $c(bd) = d(bc)$ Definition of equal rational numbers.

Thus, $\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}$ Definition of equal rational numbers and principle of substitution

Therefore, the addition of any two rational numbers is given as:

$$\boxed{\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}}$$

Example:

Add the rational numbers $\frac{7}{5}, \frac{9}{8}$

According to the definition of Addition of rational numbers:

$$\frac{7}{5} + \frac{9}{8} = \frac{(7 \times 8) + (5 \times 9)}{(5 \times 8)} = \frac{56+45}{40} = \frac{101}{40}$$

It is possible to use a different approach by noting that

$\frac{7}{5} = \frac{(7 \times 8)}{(5 \times 8)} = \frac{56}{40}$ and $\frac{9}{8} = \frac{(9 \times 5)}{(8 \times 5)} = \frac{45}{40}$ Equality of rational numbers.

Thus, $\frac{7}{5} + \frac{9}{8} = \frac{56}{40} + \frac{45}{40} = \frac{56+45}{40} = \frac{101}{40}$

MULTIPLICATION OF RATIONAL NUMBERS Although many appealing devices are used in the elementary school to develop the idea of multiplication of "fractions", for us the following definition will be necessary.

Definition: For any two rational numbers, $\frac{a}{b}$ and $\frac{c}{d}$:

$$\boxed{\frac{a}{b} \cdot \frac{c}{d} = \frac{a \times c}{b \times d}}$$

Example: $\frac{3x}{7} \cdot \frac{5y}{8} = \frac{3x \cdot 5y}{7 \cdot 8} = \frac{15xy}{56}$

According to the definition $\frac{3}{1} \cdot \frac{1}{5} = \frac{3 \times 1}{1 \times 5} = \frac{3}{5}$

since $\frac{3}{1} = 3$, we say that $\frac{3}{5} = (3 \cdot \frac{1}{5})$

Example: Multiply the following rational numbers:

(a) $\frac{2}{4} \cdot \frac{g + h}{7} = \frac{2(g + h)}{4 \cdot 7} = \frac{2g + 2H}{28}$.

(b) $\frac{5}{10} \cdot \frac{9}{18} = \frac{?}{?}$ $\quad \frac{5}{10} = \frac{1}{2}$ since $5 \times 2 = 10 \times 1$

$\frac{9}{18} = \frac{1}{2}$ since $9 \times 2 = 18 \times 1$

Therefore, $\frac{5}{10} \cdot \frac{9}{18} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

(c) Using the method of (b) multiply $\frac{4}{6} \cdot \frac{8}{4}$

POSTULATES FOR THE RATIONAL NUMBER SYSTEM

The last step in the development of the rational number system is the statement of the existance of certain axioms which must hold true for the rational numbers under the operations of addition and multiplication.

POSTULATES

(1) Closure For any two rational numbers, $\frac{a}{b}$, $\frac{c}{d}$, their sum must be a rational number. For any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, their product must be a rational number.

(2) Commutativity For any two rational numbers, $\frac{a}{b}$ and $\frac{c}{d}$

$\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$ (The Commutative Law of Addition)

For any two rational numbers, $\frac{a}{b}$ and $\frac{c}{d}$

$\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$ (Commutative Law of Multiplication)

(3) Associativity If $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ are rational numbers, then

- continued

$\frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f}$ (Associative Law of Addition)

$\frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right) = \left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f}$ (Associative Law of Multiplication)

(4) Distributive Axiom If $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ are rational numbers, then

$$\frac{a}{b} \cdot \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{ac}{bd} + \frac{ae}{bf}$$

(5) Multiplicative Identity There exists an identity element, 1, such that for all rational numbers $\frac{a}{b}$, $\frac{a}{b} \cdot 1 = \frac{a}{b}$

(6) Additive Identity There exists an identity element, 0, such that for all rational numbers $\frac{a}{b}$, $\frac{a}{b} + 0 = \frac{a}{b}$

(7) Multiplicative Inverse (Reciprocal) For all rational numbers, a, b, $a \neq 0$ and $b \neq 0$, there exists a rational number, $\frac{b}{a}$, such that $\frac{a}{b} \cdot \frac{b}{a} = 1$

Many times this postulate is stated differently as: For all rational numbers, a, there exists some rational number called the reciprocal of a (the reciprocal has the form, $\frac{1}{a}$, or a^{-1}), such that $a \cdot \frac{1}{a} = 1$, where $a \neq 0$.

(8) Additive Inverse For all rational numbers $\frac{a}{b}$, there exists the negative of $\frac{a}{b}$, $-\frac{a}{b}$, such that $\frac{a}{b} + \left(-\frac{a}{b}\right) = 0$.

CANCELLATION PROPERTIES Although cancellation properties are not always stated as postulates for the rational number system, they are helpful in problem-solving and so will be stated.

(1) Cancellation Property of Addition If $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ are rational numbers and if $\frac{a}{b} + \frac{c}{d} = \frac{e}{f} + \frac{c}{d}$ then $\frac{a}{b} = \frac{e}{f}$

(2) Cancellation Property of Multiplication If $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$ are rational numbers and $\frac{c}{d} \neq 0$ and if $\frac{a}{b} \cdot \frac{c}{d} = \frac{e}{f} \cdot \frac{c}{d}$ then $\frac{a}{b} = \frac{e}{f}$

FINDING THE BASIC MEMBER OF AN EQUIVALENCE CLASS

This topic deals with "simplifying fractions". However, the terminology, "simplify" or "reduce", is misleading, since the student is led to believe that he is somehow changing the value of the rational number.

A more logical approach is used here to explain the procedure of renaming a fraction so that it is the basic member of the equivalence class which it represents. Three principles are used in the algorithm discussed:

(1) Renaming an integer

(2) Definition of multiplication of rational numbers

(3) Multiplicative Identity for rational numbers

DEFINITION: The equivalence class $\left\{\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \ldots\right\}$ has as its basic member the rational number, 1. Note: 1 is certainly a rational number, since it can be put in the form $\frac{a}{b}$, i. e., $\frac{1}{1}$.

ALGORITHM: Consider the rational number $\frac{a}{b}$ with $a,b,c \in I$ and $b,c \neq 0$. Also, a and b have no common factor other than +1 or -1.

$\frac{ac}{bc} = \frac{a \times c}{b \times c}$	Renaming
$= \frac{a}{b} \times \frac{c}{c}$	Definition of Multiplication of rational numbers.
$= \frac{a}{b} \times 1$	$1 = \left\{\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \ldots \frac{f}{f} \ldots\right\}$ $f \neq 0$
$= \frac{a}{b}$	Multiplicative Identity

Example: Name the basic member of the equivalence class represented by the following rational numbers:

a) $\frac{14}{28}$

$\frac{14}{28} = \frac{1 \times 14}{2 \times 14}$	Renaming
$= \frac{1}{2} \times \frac{14}{14}$	Definition of Multiplication
$= \frac{1}{2} \times 1$	$\frac{14}{14} = 1$
$= \frac{1}{2}$	Multiplicative Identity

<u>Example</u>: Name the basic member of the equivalence class represented by the following rational numbers:

a, $\frac{14}{28}$ $\quad$ $\frac{14}{28} = \frac{1 \times 14}{2 \times 14}$ Renaming

$= \frac{1}{2} \times \frac{14}{14}$ Definition of Multiplication

$= \frac{1}{2} \times 1$ $\quad$ $\frac{14}{14} = 1$

$= \frac{1}{2}$ Multiplicative Identity.

<u>Exercises</u>:

(1) Multiply and find the basic form to represent the product:

a) $\frac{7}{8} \times \frac{9}{5}$

b) $\frac{-6}{5} \times \frac{-5}{8}$

c) $\frac{5y}{7t} \times \frac{6t}{8}$

(2) Add and find the basic form of the rational number which represents the sum:

a) $\frac{7}{8} + \frac{8}{9}$

b) $\frac{7z}{8} + \frac{8z}{6}$

c) $\frac{f}{g} + \frac{u}{h}$

<u>Exercises (Answers)</u>

(1)

a) $\frac{7}{8} \times \frac{9}{5} = \frac{7 \times 9}{8 \times 5} = \frac{63}{40}$

b) $\frac{-6}{5} \times \frac{-5}{8} = \frac{30}{40} = \frac{3 \times 10}{4 \times 10} = \frac{3}{4} \times \frac{10}{10} = \frac{3}{4} \times 1 = \frac{3}{4}$

c) $\frac{5y}{7t} \times \frac{6t}{8} = \frac{30yt}{56t}$

$= \frac{15y \times 2t}{28 \times 2t}$

$= \frac{15y}{28} \times \frac{2t}{2t}$

$= \frac{15y}{28} \times 1$

$= \frac{15y}{28}$

(2)

a) $\frac{7}{8} + \frac{8}{9} = \frac{63 + 64}{72} = \frac{127}{72}$

b) $\frac{7z}{8} + \frac{8z}{6} = \frac{42z + 64z}{48} = \frac{106z}{48} = \frac{53z \times 2}{24 \times 2}$

$$= \frac{53z}{24} \times \frac{2}{2}$$

$$= \frac{53z}{24} \times 1$$

$$= \frac{53z}{24}$$

c) $\frac{f}{g} + \frac{u}{h} = \frac{fh + gu}{gh}$

DENSITY OF THE RATIONAL NUMBER SYSTEM

The rational number system presents many challenges not encountered in the system of integers. Not the least of these challenges is the abstract concept of "Density".

Consider a number line which consists only of natural numbers:

0 1 2 3 4 5 6 7 8 9 10

If the question is asked, "What is the successor to the natural number 2?", the answer is obviously the natural number 3; since by definition, this is the way the natural number system is developed. To verify that there can be no natural number between a natural number and its immediate successor, consider the following theorem:

Theorem: There is no natural number between 0 and 1

Proof: The method of proof is one invented by the Greeks and carried down through the ages: "Reductio ad Absurdum"; this means to assume the opposite of what is to be proved as being true and show that this assumption leads to a contradiction.

Assume that there is some natural number, call it "n", such that

$$0 < n < 1$$

then, $n \cdot 0 < n \cdot n < n \cdot 1$ (Multiplicative Property)

thus, $0 < n^2 < n$

Therefore, we have the square of a natural number which is less than the natural number itself which is a contradiction. Since our conclusion is false, our assumption must also be false, therefore, there is no natural number between 0 and 1.

To be strictly correct the natural number line would be constructed in the following manner:

0 1 2 3 4 5 6 7 8 9 10

The line segment is not drawn since this gives the false impression of "betweenness".

To extend the number system to include the set of integers leaves a physical situation much like the number line using natural numbers.

-4 -3 -2 -1 0 1 2 3 4

RATIONAL NUMBER LINE AND DENSITY

The invention of the rational number system leads to the innovation of "betweenness", i.e., between any two rational numbers there is another rational number.

Density Theorem: Between any two rational numbers there is another rational number.

Proof: Let $\frac{a}{b}$ and $\frac{c}{d} \in R$ and $b > 0$, $d > 0$, $bd > 0$; suppose that $\frac{a}{b} < \frac{c}{d}$. Consider the rational number $\frac{(ad + bc)}{2bd}$

I.

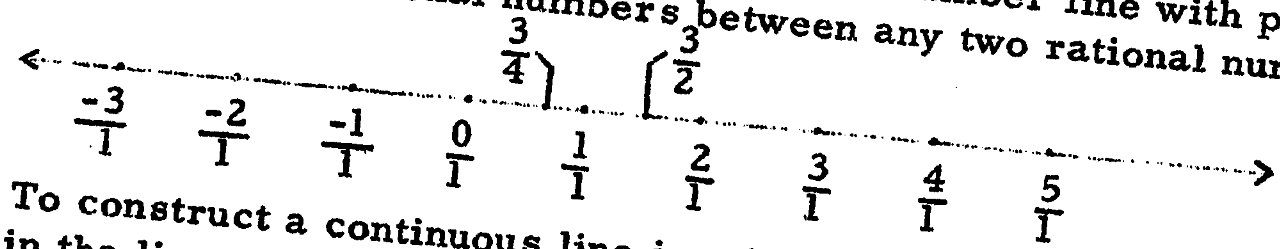
1) since $\frac{a}{b} < \frac{c}{d}$, then $ad < bc$ — Definition
2) if $ad < bc$, then $abd < b^2 c$ — Multiplicative Property
3) $2abd = abd + abd < abd + b^2 c$ — Additive Property
4) therefore, $\frac{a}{b} < \frac{(ad + bc)}{2bd}$ — Definition

II.

1) If $\frac{a}{b} < \frac{c}{d} \Longrightarrow ad < bc$ — Definition
2) If $ad < bc \Longrightarrow ad^2 < bcd$ — Multiplicative Property
3) If $ad^2 < bcd \Longrightarrow ad^2 + bcd < bcd + bcd$ — Additive Property
4) $d(ad + bc) < c(2bd)$ — Distributive Property, Commutative Property, Associative Property
5) $(ad + bc)d < (2bd)c$ — Commutative Property
6) therefore, $\frac{(ad + bc)}{2bd} < \frac{c}{d}$ — Definition

Thus the proof establishes that $\frac{a}{b} < \frac{(ad + bc)}{2bd} < \frac{c}{d}$

It is now possible to construct a rational number line with points corresponding to rational numbers between any two rational numbers.



To construct a continuous line is still inaccurate since there are gaps in the line, i.e., points on the line which do not correspond with any rational number. These gaps will correspond to what are called "Irrational Numbers". The idea of an irrational number can be understood by considering the next topic.

TERMINATING AND NON-TERMINATING DECIMALS

Through the centuries, because of the universality of base 10 and man's constant efforts to express mathematical ideas in concise form, certain fractions have been expressed in what is known as "Decimal Form". Because of the frequency with which rational numbers, such as $\frac{3}{10}$, $\frac{4}{100}$, and $\frac{23}{1000}$ appear, the use of a period, known as a decimal point, to take the place of the denominator of the fraction, has been invented.

TERMINATING DECIMAL By a Terminating Decimal is meant a decimal which may be expressed as rational number with a denominator which is a power of 10.

Example: .34 is an example of a terminating decimal since it may be written as $\frac{34}{100} = \frac{34}{10^2}$

NON - TERMINATING DECIMAL By a Non-Terminating Decimal is meant a decimal which cannot be expressed as a rational number with a power of ten as denominator.

Example: .333333333... is an example of a non-terminating decimal. Non-Terminating Decimals can be of two types, repeating and non-repeating.

NON-TERMINATING AND REPEATING DECIMALS A Non-Terminating, Repeating Decimal is a decimal which cannot be expressed as a rational number with a power of ten as a denominator, and has a certain sequence of digits which repeat.

Example: .3333333... is a non-terminating, repeating decimal with the digit "3" repeating.

.45454545... is a non-terminating, repeating decimal with the digits "45" repeating.

Although this type of decimal does not have a power of ten as a denominator, it can still be expressed in rational form through the use of a special method:

Let $a = .7777777...$

Problem: To express the number "a" in rational form, i.e.

$\frac{p}{q}$, $q \neq 0$ and $p, q \in I$

Proof:

1)	$a = .7777777777...$	
2)	$10 = 10$	Reflexive Property
3)	$10a = 7.7777777777...$	Multiplicative Property
4)	$9a = 7$	Cancellation Property with (1) and (3)

5) $\frac{1}{9} = \frac{1}{9}$ Reflexive Property

6) $\frac{1}{9}(9a) = \frac{1}{9}(7)$ Multiplicative Property

7) $(\frac{1}{9} \cdot 9)\,a = \frac{7}{9}$ Associative Property and Definition of Multiplication

8) $(1)\,a = \frac{7}{9}$ Multiplicative Inverse

9) $a = \frac{7}{9}$ Multiplicative Identity

Example:

$$a = .232323\ldots$$
$$100a = 23.23232323\ldots$$
$$99a = 23$$
$$a = \frac{23}{99}$$

Exercises:

1) Identify the following as terminating or non-terminating decimals:

(a) $\frac{4}{8}$ (b) $\frac{7}{15}$ (c) $\frac{8}{9}$ (d) $.789789\ldots$

2) Change the following to rational form:

(a) $.606$ (b) $.67$ (c) $.145145145\ldots$

NON-TERMINATING NON-REPEATING DECIMALS Decimals, which are non-terminating and do not have a digit or sequence of digits which repeat, are called Non-Terminating, Non-Repeating Decimals. Such decimals are not rational numbers and cannot, therefore, be put in rational form. It was this kind of number which was referred to when the "gaps" in the rational number line were mentioned. This kind of number is called an Irrational Number.

Exercises: (Answers)

1) (a) Terminating
(b) Non-terminating, but repeating
(c) Non-terminating, but repeating
(d) Non-terminating

2) (a) $a = .606 = \frac{606}{1000} = \frac{303}{500}$

(b) $a = .67 = \frac{67}{100}$

(c) $a = .145145145\ldots$

$1000a = 145.145145145\ldots$ Multiplicative Property

$999a = 145$ Cancellation

$a = \frac{145}{999}$ Multiplicative Inverse $\frac{1}{999}$

RATIONAL NUMBER SYSTEM ASSIGNMENT

1) Use the principles of renaming and cancellation to solve the following:

(a) $3 \times y = 15$ (b) $z + 5 = 21$ (c) $2d + 9 = 19$

2) Name the basic member of the equivalence class represented by the following rational numbers:

(a) $\frac{7}{8}$ (b) $\frac{36}{12}$ (c) $\frac{9}{12}$

3) Prove the following with the definitions given in the text:

(a) $\frac{3}{5} < \frac{7}{10}$ (b) $\frac{6}{4} \not< \frac{4}{6}$ (c) $\frac{14}{28} = \frac{1}{2}$ (d) $\frac{7}{8} > \frac{3}{5}$

4) Name the postulates which are used in the following problems:

(a) $\frac{1}{3} + \frac{2}{6} = \frac{1}{3} + \left(\frac{1}{6} + \frac{1}{6}\right)$

(b) $\frac{2}{3} \times \frac{9}{8} = \frac{9}{8} \times \left(2 \times \frac{1}{3}\right)$

(c) $\frac{5}{7} + \frac{4}{7} = \left(5 \times \frac{1}{7}\right) + \left(4 \times \frac{1}{7}\right)$

$= \left(\frac{1}{7} \times 5\right) + \left(\frac{1}{7} \times 4\right)$

$= \frac{1}{7} \times (5 + 4)$

$= \frac{1}{7} \times (9)$

$= \frac{9}{7}$

5) Name the multiplicative inverses of each of the following rational numbers:

(a) $\frac{4}{5}$ (b) $\frac{-4}{-5}$ (c) $\frac{5}{z}$ (d) 0 (e) $\frac{-9}{y}$ (f) $\frac{(x + y)}{-3}$

6) Find a rational number between the two rational numbers given:

(a) $\frac{1}{3}, \frac{1}{2}$ (b) $\frac{1}{100}, \frac{1}{1000}$ (c) $\frac{2}{3}, \frac{4}{3}$ (d) $\frac{15}{19}, \frac{15}{18}$

7) Change .172851728517285... to rational form.

RATIONAL NUMBER SYSTEM ASSIGNMENT ANSWERS

1) (a) $3 \times y = 15$
$3 \times y = 3 \times 5$ Renaming
$y = 5$ Cancellation

(b) $z + 5 = 21$
$z + 5 = 16 + 5$ Renaming
$z = 16$ Cancellation

(c) $2d + 9 = 19$
$2d + 9 = 10 + 9$ Renaming (19)
$2d = 10$ Cancellation Property of Addition
$2d = 2 \times 5$ Renaming (10)
$d = 5$ Cancellation for Multiplication

2) (a) $\frac{7}{8}$ (b) $\frac{3}{1}$ (c) $\frac{3}{4}$

3) (a) $\frac{3}{5} < \frac{7}{10} \Longleftrightarrow 30 < 35$ (b) $\frac{6}{4} \not< \frac{4}{6} \Longleftrightarrow 36 \not< 16$

(c) $\frac{14}{28} = \frac{1}{2} \Longleftrightarrow 28 = 28$ (d) $\frac{7}{8} > \frac{3}{5} \Longleftrightarrow 7 \times 5 > 8 \times 3$

4) (a) Definition of addition or renaming
(b) Commutative of multiplication and definition of multiplication
(c) c-1) Definition of multiplication or renaming
c-2) Commutative of multiplication
c-3) Distributive Law
c-4) Renaming 5 + 4
c-5) Definition of Multiplication

5) (a) $\frac{5}{4}$ (b) $\frac{-5}{-4}$ c) $\frac{z}{5}$ (d) none (e) $\frac{y}{-9}$ (f) $\frac{-3}{(x+y)}$

6) (a) $\frac{5}{12}$ (b) $\frac{2}{1000}$ (c) $\frac{3}{3}$ (d) $\frac{276}{342}$

7) Rational form: $\frac{17285}{99,999}$